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Existence and uniqueness of solutions for a class of piecewise linear dynamical systems

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Abstract

We consider the class of dynamical systems that arises when inputs and outputs of a linear system are connected pairwise by means of piecewise linear algebraic relations. It is not assumed that these relations define inputs in terms of outputs or vice versa; in particular, the relations need not be Lipschitzian. We obtain conditions for existence and uniqueness of solutions of such dynamical systems in the class of piecewise Bohl functions. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

The consideration of dynamical systems with external variables can be motivated in several ways. In control theory, external variables occur as actuators and sensors. In a hierarchical modeling context, external variables arise as the variables through which subsystems may be connected to each other. In studies of dissipative systems, inputs and outputs are used in pairs to quantify energy exchange. Some interesting classes of systems may be obtained by connecting inputs and outputs to each other

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in a particular way; for instance, letting certain inputs and outputs be linked by a feedback with given maximal L_2-gain has been popular in recent years as a way of describing model uncertainty. The many uses that can be made of inputs and outputs (or more generally, of external variables) show the strength of systems theory as it has been developed in the past decades.

In this paper we will be concerned with yet another way of using external variables. Similar to the way that inputs and outputs are used in the model uncertainty description that we just mentioned, the class of systems that we discuss below is obtained by connecting inputs and outputs in a specific way. Similar to the use of inputs and outputs in the context of dissipativity, the links that we specify are defined for pairs of (scalar) inputs and outputs. Below we consider the class of dynamical systems that is obtained by combining a dynamic linear input/output system with a static piecewise linear input/output relation. The properties of the systems that are obtained in this way are coded entirely into the usual (A, B, C, D) parameters of the linear system and the parameters of the piecewise linear relation. As a result, we study in this paper a class of nonlinear and nonsmooth dynamical systems using notions from linear systems theory.

Although of course many properties of the systems considered here are of potential interest, we concentrate in this paper on the most basic properties, namely, existence and uniqueness of solutions. As already mentioned, we consider piecewise linear relations between pairs of variables. These relations do not necessarily specify one variable as a function of the other (see e.g. the characteristics shown in Fig. 1 or Fig. 3(b) below). For this reason, standard theorems on the well-posedness of feedback connections do not apply. Nevertheless, it will be shown below that existence and uniqueness of solutions do hold if certain conditions are satisfied. Of course one may consider piecewise linear systems with additional inputs and outputs that are not connected by piecewise linear relations; here however we study the basic situation in which there are no additional external variables so that solutions, if uniqueness holds, are parametrized by initial conditions.

The systems that we consider can also be studied in the framework of differential inclusions, as developed for instance in [1]. Indeed this is a very general framework that allows the study of many kinds of systems. We believe that the approach taken in this paper is more tailored to the specific structure of piecewise linear systems; as will be shown below, some ideas from linear systems theory actually play an important role in the formulation of necessary and sufficient conditions for existence and uniqueness of solutions. The work by Filippov (see for instance [10]) is closer to the approach we take here than the general differential inclusion framework. On the one hand discontinuous dynamical systems in Filippov's sense may be described in terms of relays, which are a special case of the piecewise linear characteristics considered here; on the other hand Filippov allows nonlinear dynamics, whereas we restrict ourselves to couplings between piecewise linear characteristics and linear dynamics. As a result we obtain conditions for well-posedness that are different in nature from those considered by Filippov.

Piecewise linear systems are important for several reasons:

- They form a limited class which nevertheless can approximate nonlinear phenomena as accurately as desired.
- As quite natural extensions of linear systems, they allow already well-established linear analysis/synthesis methods to be applied locally.
- They arise naturally in many applications ranging from circuit theory to economics and from mechanics to control systems.

To give a quick impression of application areas, we mention linear electrical circuits with piecewise resistive elements [2,19,29], systems with relays [27] and/or saturation characteristics, mechanical systems with Coulomb friction [21], variable structure systems [28], and bang–bang control [3,18].

In many of the application areas mentioned, one encounters piecewise linear relations between two variables that cannot be rewritten as functions from one variable to the other. For instance, the relay characteristic is of such a type. Although it would be possible to apply a change of coordinates so as to obtain a functional relationship (for instance, rotation by 45° in the relay example), such a transformation will affect the feedthrough term in the linear system component of the overall system description. As is well-known, even Lipschitzian feedback may not be well-posed for linear systems with feedthrough terms. In the development below, we allow piecewise linear relations of non-functional type as well as nonzero feedthrough terms in the part of the system that is specified by parameters (A, B, C, D).

As is well-known (see for instance [9]), piecewise linear relations may be described in terms of the linear complementarity problem (LCP) of mathematical programming. The LCP is briefly described in Section 4 below, together with one of its generalizations, the *horizontal* complementarity problem (HLCP). The complementarity formulation has been used for *static* piecewise linear systems in [19,29]; the present paper may be viewed as an extension of the cited work in the sense that we consider *dynamic* systems. The paper can also be viewed as a generalization of earlier work which was concerned with well-posedness of linear systems coupled to the ideal diode (pure complementarity) characteristic [15,24] or to the relay characteristic [20], although the approach taken here is somewhat different from the one in [20].

The organization of the paper is as follows. We begin with a quick look at motivational examples in Section 2. Section 3 is devoted to the introduction of the piecewise linear characteristics that will be under investigation in the sequel. This will be followed by recalling the related complementarity problems in Section 4. In Section 5, we propose a definition of the notion of solution for linear systems with piecewise linear characteristics. We then first give conditions for existence and uniqueness of solutions locally in time in Section 6, and we proceed to discuss global solutions in Section 7. The results that we obtain can be specialized to give well-posedness results for several classes of systems, including linear systems with relays or saturation characteristics; this is shown in Section 8. Conclusions follow in Section 9. There is one appendix (Appendix A) containing a technical point relating to the Lipschitzian dependence on data of solutions to the HLCP.

The following notational conventions will be in force. The symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}(s)$ and $\mathbb{C}$ denote the sets of real numbers, nonnegative real numbers, real coefficient rational functions and complex numbers, respectively. For a given integer *n*, we write $\overline{n}$ for the set $\{1, 2, \dots, n\}$. Let $\mathscr{X}$ be a set. The notations $\mathscr{X}^n$ and $\mathscr{X}^{n\times m}$ where *n* and *m* are integers denote the sets of *n*-tuples and $n \times m$ matrices of the elements of $\mathscr{X}$. The set of subsets of $\mathscr{X}$ will be denoted by $2^{\mathscr{X}}$. We write $|\mathscr{X}|$ for the number of elements of $\mathscr{X}$. Let $A \in \mathscr{X}^{n\times m}$ be a matrix of the elements of the set $\mathscr{X}$. We write A_{ij} for the (i, j)th element of *A*. The transpose of *A* is denoted by A^{T}. For $J \subseteq \overline{n}$, and $K \subseteq \overline{m}$, A_{JK} denotes the submatrix $\{A_{ij}\}_{j\in J, i\in K}$. If $J = \overline{n}$ ($K = \overline{m}$), we also write $A_{\bullet K}$ ($A_{J\bullet}$). In order to avoid bulky notation, we use A^{T}_{JK} and A^{-1}_{JK} instead of (A^{T}_{JK}) and $(A_{JK})^{-1}$, respectively. Given two matrices $A \in \mathscr{X}^{n_a\times m}$ and $B \in \mathscr{X}^{n_b\times m}$, the matrix obtained by stacking *A* over *B* is denoted by $\mathrm{col}(A, B)$. The diagonal matrix with the diagonal elements $a_1, a_2, \dots, a_n$ is denoted by $\mathrm{diag}(a_1, a_2, \dots, a_n)$. A rational matrix $A(s) \in \mathbb{R}^{n\times m}(s)$ is said to be *proper* if $\lim_{s\to\infty} A(s)$ is finite. If $\lim_{s\to\infty} A(s) = 0$, it is said to be *strictly proper*. We denote ordered sets by $[\cdots]$ and interior of a set by $(\cdot)^\circ$. For a given set $\mathscr{S} \subset \mathbb{R}^n$, $\mathrm{affn}\,\mathscr{S}$ denotes its affine hull. We say that a proposition $\mathscr{P}(x)$ holds for all sufficiently small (large) *x* if there exists $x_0 > 0$ such that it holds for all $0 \leqslant x \leqslant x_0$ ($x_0 \leqslant x$).

2. Motivational examples

In circuit theory, piecewise linear modeling is a widely used technique. For instance, ideal modeling of a diode yields a voltage–current characteristic depicted in Fig. 1. Similar-looking characteristics can be obtained from parallel/series connections of linear resistors, ideal diodes and batteries. Such a circuit and its voltage–current characteristic are shown in Fig. 2. We can think of many other piecewise resistive elements such as saturation characteristics (see Fig. 3) or dynamical elements such as capacitors/inductors with piecewise linear charge-voltage/flux-current characteristics. Of course, piecewise linear elements also occur in various other engineering areas. For instance, the ideal relay characteristic (see Fig. 3) serves as an idealized model of Coulomb friction in mechanical systems and it arises as well in switching control schemes. Many other examples and potential application areas of piecewise linear phenomena can be found. With these wide-range application areas in our mind, we will address the well-posedness (in the sense of existence and uniqueness of

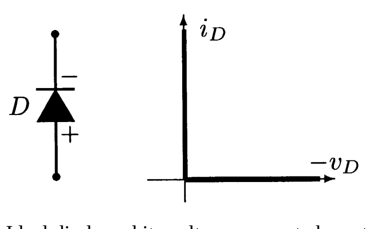

Fig. 1. Ideal diode and its voltage–current characteristic.

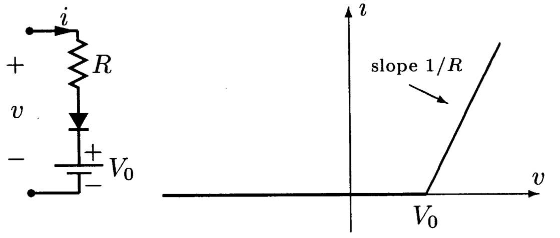

Fig. 2. A piecewise linear resistor and its voltage–current characteristic.

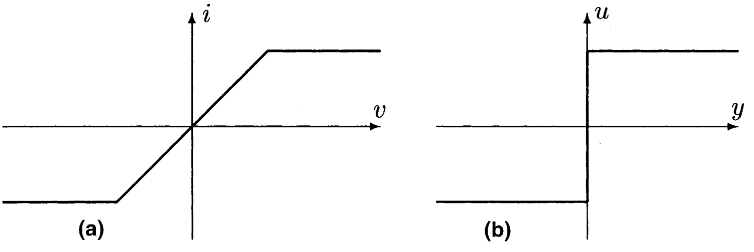

Fig. 3. Saturation and ideal relay characteristics: (a) saturation characteristic; (b) ideal relay characteristic.

solutions) issues of models consisting of a linear (dynamical) system coupled with elements ($\mathscr{G}^i$'s) that are of a piecewise linear nature (see Fig. 4). The piecewise linear elements that we consider in the paper are 2-dimensional piecewise linear curves. These curves are not necessarily the graph of a function that is defined from $\mathbb{R}$ to $\mathbb{R}$ (see ideal diode and relay characteristics above). In general, they are relations on $\mathbb{R} \times \mathbb{R}$. We say that a 2-dimensional piecewise linear curve is *function-like* if it coincides with the graph of a function that is defined from $\mathbb{R}$ to $\mathbb{R}$. For instance, the saturation characteristic is function-like. It is known from the theory of ordinary differential equations that if the piecewise linear curves $\mathscr{G}^i$ are function-like and the linear system has no feedthrough term (i.e. $D = 0$) then the existence and uniqueness

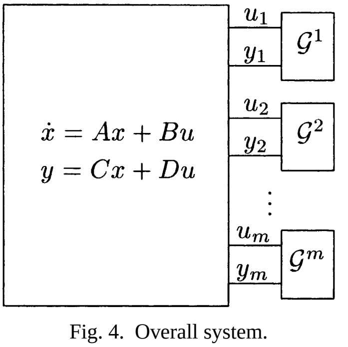

Fig. 4. Overall system.

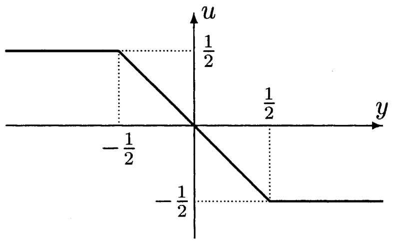

Fig. 5. Saturation characteristic.

of the solutions of the overall system (see Fig. 4) follows from a Lipschitz continuity argument. However, the presence of a nonzero feedthrough term makes it possible to find *ill-posed* examples as illustrated in the following example.

Example 2.1. Consider the single-input, single-output system

$$\dot{x} = u, \tag{1a}$$

$$y = x - 2u, \tag{1b}$$

where u and y are restricted by a saturation characteristic shown in Fig. 5. Let the periodic function $\tilde{u} : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$\tilde{u}(t) = \begin{cases} 1/2 & \text{if } 0 \leqslant t < 1, \\ -1/2 & \text{if } 1 \leqslant t < 3, \\ 1/2 & \text{if } 3 \leqslant t < 4 \end{cases}$$

and $\tilde{u}(t-4) = \tilde{u}(t)$ whenever $t \geqslant 4$. By using this function define $\tilde{x} : \mathbb{R}_+ \to \mathbb{R}$ as

$$\tilde{x}(t) = \int_0^t \tilde{u}(s)\,\mathrm{d}s,$$

and $\tilde{y} : \mathbb{R}_+ \to \mathbb{R}$ as

$$\tilde{y} = \tilde{x} - 2\tilde{u}.$$

It can be verified that $(-\tilde{u}, -\tilde{x}, -\tilde{y})$, $(0, 0, 0)$, and $(\tilde{u}, \tilde{x}, \tilde{y})$ all satisfy (1a) and (1b). Moreover, $(-\tilde{u}, -\tilde{y})$, $(0, 0)$, and $(\tilde{u}, \tilde{y})$ all lie on the saturation characteristic.

3. Piecewise linear characteristics and their representations

The main ingredients of this section are *piecewise linear characteristics*. We consider only those characteristics which are piecewise affine curves in $\mathbb{R}^2$ as it is defined in the following.

Definition 3.1. A set $\mathscr{G}$ is called a *k-piecewise linear characteristic* if there exist (*directions*) $d^-, d^+ \in \mathbb{R}^2$ with $\|d^-\| = \|d^+\| = 1$ and (*vertices*) $[v^i]_{i=1}^{k-1} \in (\mathbb{R}^2)^k$ such that the two half lines

$$\mathscr{G}_1 = \left\{\lambda d^- + v^1 \mid 0 \leqslant \lambda\right\}, \tag{2a}$$

$$\mathscr{G}_k = \left\{v^{k-1} + \lambda d^+ \mid 0 \leqslant \lambda\right\} \tag{2b}$$

and $k-2$ line segments

$$\mathscr{G}_i = \left\{\lambda v^{i-1} + (1-\lambda)v^i \mid 0 \leqslant \lambda \leqslant 1\right\} \quad \text{for } i = 2, 3, \ldots, k-1 \tag{2c}$$

satisfy the following conditions
1. $\mathscr{G}_i \cap \mathscr{G}_{i+1} = \{v_i\}$ for $i = 1, 2, \ldots, k-1$,
2. $\mathscr{G}_i \cap \mathscr{G}_j = \emptyset$ if $|i-j| > 1$,
3. $\bigcup_{i=1}^k \mathscr{G}_i = \mathscr{G}$.

If the above conditions hold we write $\mathscr{G} = \mathrm{plc}(d^-, [v^i]_{i=1}^k, d^+)$. We say that $(d^-, [v^i]_{i=1}^k, d^+)$ is a *minimal description* of $\mathscr{G}$ if $\mathscr{G} = \mathrm{plc}(d^-, [v^i]_{i=1}^k, d^+)$ and $\mathscr{G}$ is not a $(k-1)$-piecewise linear characteristic. We say that the vertex $v \in [v^i]_{i=1}^k$ of $\mathrm{plc}(d^-, [v^i]_{i=1}^k, d^+)$ is *redundant* if $\mathrm{plc}(d^-, [v^i]_{i=1}^k \backslash v, d^+) = \mathrm{plc}(d^-, [v^i]_{i=1}^k, d^+)$.

Remark 3.2. Notice that

$$\mathrm{plc}(d^-, [v^1, v^2, \ldots, v^k], d^+) = \mathrm{plc}(d^+, [v^k, v^{k-1}, \ldots, v^1], d^-),$$

i.e., the d's and v's are not unique. Notice also that every k-piecewise linear characteristic can be regarded as a $k+p$-piecewise linear characteristic by adding p redundant vertices. It can be verified that there are exactly two minimal descriptions for every $\mathscr{G}$.

An example of a k-piecewise linear characteristic is depicted in Fig. 6. It is known that (see e.g. [9,17,29]) such piecewise linear curves can be represented by using *complementarity* variables. In [9], the *conceptual* equivalence of piecewise linear functions and linear complementarity problems is shown. However, the LCP obtained as an equivalent of a piecewise linear function is of special form for which the available theory of LCP cannot yield direct implications. A similar type of equivalence is addressed in [29]. The paper considers not only piecewise linear functions

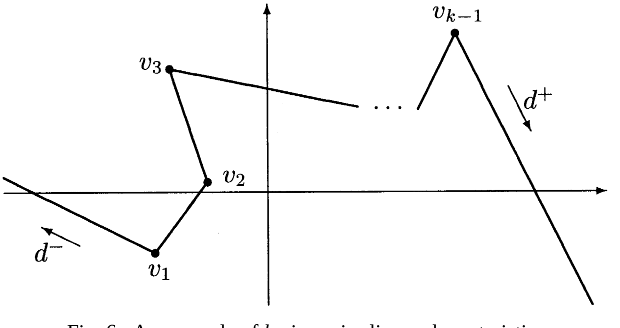

Fig. 6. An example of k-piecewise linear characteristic.

but also piecewise linear *relations* in general. However, what has been shown is that such characteristics are equivalent to generalized linear complementarity problems for which there are no known conditions for (unique) solvability. Our approach is rather close to the one that has been taken in [17]. Kaneko showed there that piecewise linear functions are equivalent to the functions defined by so-called *horizontal linear complementarity problem* (HLCP). The existence and uniqueness of the solutions of HLCP have been studied in [17,26]. We shall first extend Kaneko's equivalence to piecewise linear relations. Then, our treatment will heavily rely on this equivalence and the study on HLCP.

The next definition is a first step towards introducing *complementarity representations* for k-piecewise linear characteristics.

Definition 3.3. An ordered set $[z^i]_{i=1}^k \in (\mathbb{R}^m)^k$ is called *k-horizontal complementary* if the following conditions hold:

$$0 \leqslant z^1, \tag{3}$$

$$0 \leqslant z^i \leqslant e \quad \text{for } i = 2, 3, \ldots, k-1, \tag{4}$$

$$0 \leqslant z^k, \tag{5}$$

$$(z^1)^{\mathrm{T}} z^2 = 0, \tag{6}$$

$$(e - z^i)^{\mathrm{T}} z^{i+1} = 0 \quad \text{for } i = 2, 3, \ldots, k, \tag{7}$$

where e denotes the vector of ones. The set of all such k-horizontal complementary ordered sets is denoted by $\mathscr{HC}_k^m$. For the sake of brevity, we will write $\mathscr{HC}_k$ instead of $\mathscr{HC}_k^1$.

We will often use the following particular description of the set $\mathscr{HC}_k$.

Proposition 3.4. *Let the sets $[\zeta^i]_{i=1}^k$ be defined as*

$$\zeta^1 = \left\{[z^i]_{i=1}^k \in \mathbb{R}^k \mid 0 \leqslant z^1,\ z^2 = z^3 = \cdots = z^k = 0\right\}, \tag{8a}$$

$$\zeta^k = \left\{[z^i]_{i=1}^k \in \mathbb{R}^k \mid 0 \leqslant z^k, z^1 = 0,\ z^2 = z^3 = \cdots = z^{k-1} = 1\right\}, \tag{8b}$$

and for $j = 2, 3, \ldots, k-1$,

$$\zeta^j = \left\{[z^i]_{i=1}^k \in \mathbb{R}^k \;\middle|\; 0 \leqslant z^j \leqslant 1 \text{ and } z^i = \begin{cases} 0, & i = 1, \\ 1, & i = 2, 3, \ldots, j-1, \\ 0, & i = j+1, j+2, \ldots, k. \end{cases}\right\}. \tag{8c}$$

Then the following statements hold:
1. *For $i = 1, 2, \ldots, k-1$, $\zeta^i \cap \zeta^{i+1}$ is a singleton.*
2. $\zeta^i \cap \zeta^j = \emptyset$ *if* $|i - j| > 1$.
3. $\bigcup_{i=1}^k \zeta^i = \mathscr{HC}_k$.

The proof of the above proposition directly follows from the definitions of the sets ζ^j.

There is a correspondence between k-piecewise linear characteristics and affine functions defined on the set $\mathscr{HC}_k$. To see this, consider a k-piecewise linear characteristic $\mathscr{G} = \mathrm{plc}(d^-, [v^1, v^2, \dots, v^k], d^+)$ and the affine function $f : \mathscr{HC}_k \to \mathbb{R}^2$ given by

$$f([z^i]_{i=1}^k) := v^1 + d^- z^1 + \sum_{j=2}^{k-1}(v^j - v^{j-1})z^j + d^+ z^k. \tag{9}$$

Let the sets $[\zeta^i]_{i=1}^k$ be as in Proposition 3.4. Note that $f(\zeta^i) = \mathscr{G}_i$. Moreover, it can be verified that f is a bijection. We will represent piecewise linear characteristics by exploiting this correspondence. With this aim, consider m k-piecewise linear characteristics $[\mathscr{G}^i]_{i=1}^m$. We associate to each characteristic $\mathscr{G}^i = \mathrm{plc}(d^{i,-}, [v^{i,1}, v^{i,2}, \dots, v^{i,k}], d^{i,+})$ two vectors

$$r^i = \mathrm{col}\big(-d_1^{i,-}, v_1^{i,2} - v_1^{i,1}, \dots, v_1^{i,k-1} - v_1^{i,k-2}, d_1^{i,+}\big), \tag{10a}$$

$$s^i = \mathrm{col}\big(-d_2^{i,-}, v_2^{i,2} - v_2^{i,1}, \dots, v_2^{i,k-1} - v_2^{i,k-2}, d_2^{i,+}\big). \tag{10b}$$

and a function $f^i : \mathscr{HC}_k \to \mathbb{R}^2$ defined by

$$f^i([z^i]_{i=1}^k) := v^{i,1} - \begin{pmatrix} r_1^i \\ s_1^i \end{pmatrix} z^1 + \begin{pmatrix} r_2^i \\ s_2^i \end{pmatrix} z^2 + \begin{pmatrix} r_3^i \\ s_3^i \end{pmatrix} z^3 + \cdots + \begin{pmatrix} r_k^i \\ s_k^i \end{pmatrix} z^k. \tag{11}$$

Define $q^u, q^y \in \mathbb{R}^m$ as $q^u = \mathrm{col}(v_1^{1,1}, v_1^{2,1}, \dots, v_1^{m,1})$ and $q^y = \mathrm{col}(v_2^{1,1}, v_2^{2,1}, \dots, v_2^{m,1})$. Also define $[R^j]_{j=1}^k, [S^j]_{j=1}^k \in (\mathbb{R}^{m\times m})^k$ as $R^j = \mathrm{diag}(r_j^1, r_j^2, \dots, r_j^m)$ and $S^j = \mathrm{diag}(s_j^1, s_j^2, \dots, s_j^m)$.

Fact 3.5. *Consider m k-piecewise linear characteristics $[\mathscr{G}^i]_{i=1}^m$. Let $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ be as defined above. Then, the following statements are equivalent.*

1. *For each $i \in \overline{m}$,*

$$\begin{pmatrix} u_i \\ y_i \end{pmatrix} \in \mathscr{G}^i. \tag{12}$$

2. *For some $[z^i]_{i=1}^k \in \mathscr{HC}_k^m$,*

$$u = q^u - R^1 z^1 + R^2 z^2 + R^3 z^3 + \cdots + R^k z^k, \tag{13a}$$
$$y = q^y - S^1 z^1 + S^2 z^2 + S^3 z^3 + \cdots + S^k z^k. \tag{13b}$$

Moreover, the mapping $\mathrm{col}(u, y) \mapsto [z^i]_{i=1}^k$ *is a bijection.*

Indeed, the assertion follows immediately from the fact that each f^i is a bijection.

Definition 3.6. We say that $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ is a *horizontal complementarity representation* of $[\mathscr{G}^i]_{i=1}^m$.

It is clear from the discussion following Definition 3.1 that these representations are not unique.

4. Complementarity problems

Our treatment will be based on the complementarity problems of mathematical programming. In this section, we briefly recall complementarity problems in order to be self-contained. We begin with the linear complementarity problem (LCP). The book [8] is an excellent survey on the LCP.

Problem 4.1 (LCP(q, M)). Given $q \in \mathbb{R}^m$ and $M \in \mathbb{R}^{m\times m}$, find $z \in \mathbb{R}^m$ such that

$$z \geqslant 0, \tag{14a}$$

$$q + Mz \geqslant 0, \tag{14b}$$

$$z^{\mathrm{T}}(q + Mz) = 0. \tag{14c}$$

We say that z is *feasible* if it satisfies (14a) and (14b). Similarly, we say *z solves* LCP(q, M) if it satisfies (14a)–(14c). The set of all solutions of LCP(q, M) will be denoted by SOL(q, M). In general, SOL(q, M) may be empty. The notation $K(M)$ denotes the set $\{q \mid \mathrm{SOL}(q, M) \neq \emptyset\}$. It is easy to see that $\mathbb{R}^m{}_+ \subseteq K(M)$ for all $M \in \mathbb{R}^{m\times m}$. The following fact on the closedness of $K(M)$ will be used several times in the sequel.

Fact 4.2. *The set $K(M)$ is closed for any matrix M.*

The LCP leads to the study of a substantial number of matrix classes that relate to several aspects of the problem such as feasibility, solvability, unique solvability. The following ones will be of particular interest for our purposes.

Definition 4.3. A matrix $M \in \mathbb{R}^{m\times m}$ is called
- *nondegenerate* if all its principal minors are nonzero,
- *positive* (*nonnegative*) *definite* if $x^{\mathrm{T}}Mx > 0$ ($\geqslant 0$) for all $0 \neq x \in \mathbb{R}^m$,
- a *$\mathscr{P}$-matrix* if all its principal minors are positive.

Note that every positive definite matrix is $\mathscr{P}$-matrix. The class $\mathscr{P}$ plays an important role in the study of the LCP as the following standard result indicates.

Theorem 4.4 ([8, Theorem 3.3.7]). *Let $M \in \mathbb{R}^{m\times m}$ be given. Then* LCP(q, M) *has a unique solution for all $q \in \mathbb{R}^m$ if and only if M is a $\mathscr{P}$-matrix.*

There are a number of interesting generalizations of the LCP of mathematical programming. Particularly, the (Extended) Horizontal LCP will play a key role in representing piecewise linear characteristics.

Problem 4.5 (HLCP$(q, [M^i]_{i=1}^k)$). Given $q \in \mathbb{R}^m$ and $[M^i]_{i=1}^k \in \mathbb{R}^{m\times m}$, find $[z^i]_{i=1}^k \in \mathscr{HC}_k^m$ such that

$$M^1 z^1 = q + \sum_{i=2}^{k} M^i z^i.$$

The HLCP was introduced in [16] with $k = 3$ and $M^1 = I$, and further developed in [17] with an eye towards piecewise linear functions.

We briefly recall some facts from [26] and state a result on solvability of the problem which is parallel to Theorem 4.4. To do this we need to state some definitions.

Definition 4.6. A matrix $R \in \mathbb{F}^{m\times m}$ is called *a column representative* of $[M^i]_{i=1}^k \in (\mathbb{F}^{m\times m})^k$ if

$$R_{\bullet i} \in \left\{M^1_{\bullet i}, M^2_{\bullet i}, \dots, M^k_{\bullet i}\right\} \quad \text{for all } i \in \overline{m}.$$

For a given $l \in \overline{k}^m$, the matrix $([M^i]_{i=1}^k)^l$ is defined by

$$([M^i]_{i=1}^k)^l_{\bullet j} = M^{l_j}_{\bullet j} \quad \text{for } j = 1, 2, \dots, m.$$

In the sequel, we follow the terminology of [26].

Definition 4.7. We say that an ordered set of matrices $[M^i]_{i=1}^k$
- is *nondegenerate* if all column representative matrices are nondegenerate,
- [26] has the *column $\mathscr{W}$-property* if the determinants of the column representative matrices are either all positive or all negative.

For the sake of completeness, we quote the following theorem from [26].

Theorem 4.8 [26]. HLCP$(q, [M^i]_{i=1}^k)$ *has a unique solution for all $q \in \mathbb{R}^m$ if and only if $[M^i]_{i=1}^k$ has the column $\mathscr{W}$-property.*

Remark 4.9. The LCP can be regarded as a special case of the HLCP. Indeed, LCP(q, M) is nothing but HLCP$(q, [I, M])$. In this case, Theorems 4.4 and 4.8 coincide since M is a $\mathscr{P}$-matrix if and only if the determinants of all column representative matrices of $[I, M]$ are positive, i.e., $[I, M]$ has the column $\mathscr{W}$-property. On the other hand, HLCP$(q, [M^i]_{i=1}^k)$ can be written as an LCP whenever M^1 is invertible. For this purpose, we define

$$r(q) := \mathrm{col}(q, e, e, \dots, e),$$

$$N([M^i]_{i=1}^k) := \begin{pmatrix} \tilde{M}^1 & \tilde{M}^2 & \cdots & \tilde{M}^{k-1} \\ -I & 0 & \cdots & 0 \\ 0 & -I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -I \end{pmatrix},$$

where $\tilde{M}^i = (M^1)^{-1}M^{i+1}$ for $i = \overline{k-1}$. There is a one-to-one correspondence between the solutions of HLCP(q, $[M^i]_{i=1}^k$) and LCP($r((M^1)^{-1}q)$, $N([M^i]_{i=1}^k)$). In fact, if $[z^i]_{i=1}^k$ solves the former then col$(z^2, z^3, \ldots, z^k)$ solves the latter and vice versa. Note however that a general solvability result like Theorem 4.4 does not apply as such because the HLCP corresponds to an LCP with a data vector of a very special form.

5. Piecewise linear systems

Consider a continuous-time, linear, time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{15a}$$

$$y(t) = Cx(t) + Du(t), \tag{15b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$ and A, B, C, and D are matrices with appropriate sizes. We denote (15a) and (15b) by $\Sigma(A, B, C, D)$. Let $[\mathscr{G}^i]_{i=1}^m$ be a given family of k-piecewise linear characteristics. Let the variables u and y be coupled via these characteristics as depicted in Fig. 4, i.e.,

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \mathscr{G}^i \tag{16}$$

for all t. We denote the resulting piecewise linear system ($\Sigma(A, B, C, D)$ together with (16) by PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$.

One way of looking at PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ is to consider it as a hybrid system (see e.g. [25]). Very roughly speaking, a hybrid system is a collection of *modes* and *mode transition rules*. Every mode has its own dynamics. The time evolution of a hybrid system consists of cycles of smooth continuations generated by the active mode dynamics and of mode transitions. More precisely, starting in a mode the trajectories of the system follow the corresponding dynamics until the mode transition rules force a mode change (called an *event*). After an event occurs the system evolves in another mode and so on. For the specific class of systems PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$, one can distinguish k^m modes. The dynamics of a mode $l \in \overline{k}^m$ can be given by the linear differential algebraic equations:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{17a}$$

$$y(t) = Cx(t) + Du(t), \tag{17b}$$

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \operatorname{affn} \mathscr{G}^i_{l_i} \quad \text{for each } i = 1, 2, \ldots, m. \tag{17c}$$

This dynamics is active as long as the condition

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \mathscr{G}^i_{l_i} \quad \text{for each } i = 1, 2, \ldots, m \tag{18}$$

holds. Our solution concept for PLS will be built on this 'hybrid system' thinking.

6. Initial solutions and their characterizations

We will employ *initial solutions*, which satisfy (17a)–(17c) globally and (18) locally (i.e. initially), to construct solutions to PLS. First, we need to introduce some nomenclature. The functions of the form $t \mapsto He^{Ft}G$, where F, G, and H are matrices with appropriate dimensions, are called *Bohl* functions after the Latvian mathematician Piers Bohl (1865–1921). They coincide with the continuous functions having rational Laplace transformation.

Definition 6.1. A triple $(u, x, y) \in \mathscr{B}^{m+n+m}$ is said to be an *initial solution* of $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x_0 if the following conditions hold:

1. It satisfies

$$\dot{x} = Ax + Bu, \quad x(0) = x_0,$$
$$y = Cx + Du.$$

2. For each $i = 1, 2, \ldots, m$,

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \mathscr{G}^i \quad \text{for all sufficiently small } t.$$

We will often use the following definition in the sequel.

Definition 6.2. The family of m-tuples of continuous functions $[f^i]_{i=1}^n$ is said to be *initially k-complementary* if the following conditions hold:

1. For all sufficiently small t,

$$0 \leqslant f^1(t),$$
$$0 \leqslant f^i(t) \leqslant e \quad \text{for } i = 2, 3, \ldots, n-1,$$
$$0 \leqslant f^n(t).$$

2. For all $t \in \mathbb{R}_+$,

$$(f^1(t))^{\mathrm{T}} f^2(t) = 0,$$
$$(e - f^i(t))^{\mathrm{T}} f^{i+1}(t) = 0 \quad \text{for } i = 2, 3, \ldots, n.$$

In the next section, we will go from initial solutions to global solutions. In this process we need a uniform continuity property of solutions; this will follow from the lemma below.

Lemma 6.3. *Consider a piecewise linear system* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$. *Let* $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ *be a horizontal complementarity representation of the piecewise linear characteristics* $[\mathscr{G}^j]_{j=1}^m$. *Also let* $G(s) = D + C(sI - A)^{-1}B$ *denote the transfer matrix of* $\Sigma(A, B, C, D)$. *Assume that all column representatives of* $[G(s)R^j - S^j]_{j=1}^k$ *are invertible as a rational matrix and the triple* (u, x, y) *is an initial solution of* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ *with some initial state. Then the following statements hold*:

1. *Let* $[\mathscr{G}^i_j]_{j=1}^k$ *be as in Definition* 3.1 *for each* $i = 1, 2, \ldots, m$. *Then, there exists* $l \in \overline{k}^m$ *such that*

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \operatorname{affn} \mathscr{G}^i_{l_i} \quad \text{for each } i = 1, 2, \ldots, m \text{ and } t \in \mathbb{R}_+.$$

2. *Let* $l \in \overline{k}^m$ *be as in the previous item.*
   (a) *There exist vectors* $\bar{u}^l, \bar{y}^l \in \mathbb{R}^m$ *and* $z \in \mathscr{B}^m$ *such that*
   $u = \bar{u}^l + \mathscr{R}^l z,$
   $y = \bar{y}^l + \mathscr{S}^l z,$
   where $\mathscr{R} = [-R^1, R^2, R^3, \ldots, R^k]$ *and* $\mathscr{S} = [-S^1, S^2, S^3, \ldots, S^k]$.
   (b) *There exist initially k-complementary Bohl functions* $[z^j]_{j=1}^k \subset \mathscr{B}^m$ *such that*
   $u = q^u - R^1z^1 + R^2z^2 + R^3z^3 + \cdots + R^kz^k,$
   $y = q^y - S^1z^1 + S^2z^2 + S^3z^3 + \cdots + S^kz^k.$
   (c) *There exist matrices* $F^l \in \mathbb{R}^{n\times n}$ *and* $G^l \in \mathbb{R}^{m\times n}$, *and vectors* $v^l \in \mathbb{R}^n$ *and* $w^l \in \mathbb{R}^m$ *depending only on l such that*
   $\dot{x} = F^l x + v^l,$
   $u = G^l x + w^l.$
   (d) *For a given* $T > 0$, *there exists* α^l *depending only on l and T such that*
   $$\|x(t) - x(s)\| \leqslant \alpha^l \|t - s\|$$
   for all $t, s \in [0, T]$.

To prove Lemma 6.3, we need some preparations.

Proposition 6.4. *Let* $\mathscr{G} \subset \mathbb{R}^2$ *be an affine set. There exist real numbers* α, β, *and* γ *such that*

$$\begin{pmatrix} v \\ w \end{pmatrix} \in \mathscr{G} \quad \Leftrightarrow \quad \alpha v + \beta w + \gamma = 0.$$

Proof. Evident. □

Lemma 6.5. *Consider a matrix quadruple* (A, B, C, D) *such that the transfer matrix* $D + C(sI - A)^{-1}B$ *is invertible as a rational matrix. Suppose that the function pair* (u, x), *where* x *is differentiable, satisfies*

$$\dot{x} = Ax + Bu + e, \tag{19a}$$

$$0 = Cx + Du + f \tag{19b}$$

for some $e \in \mathbb{R}^n$ *and* $f \in \mathbb{R}^m$. *Then,* x *is uniquely determined, and there exist a matrix* $K \in \mathbb{R}^{m\times n}$ *and a vector* $l \in \mathbb{R}^m$ *both depending only on* (A, B, C, D, e, f) *such that*

$$u = Kx + l.$$

Proof. It follows from [11, Theorem 3.24]. □

Proof of Lemma 6.3. 1. Since they are Bohl functions, both u_i and y_i are continuous. It follows from Definition 6.1 item 2 together with continuity that for each $i \in \overline{m}$ there exists $l_i \in \overline{k}$ such that

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \mathscr{G}^i_{l_i} \quad \text{for all } t \in [0, \epsilon)$$

for some $\epsilon > 0$. Since $\mathscr{G}^i_{l_i} \subseteq \operatorname{affn} \mathscr{G}^i_{l_i}$, we have

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \operatorname{affn} \mathscr{G}^i_{l_i} \quad \text{for all } t \in [0, \epsilon).$$

Then, it follows from Proposition 6.4 that for each $i = 1, 2, \ldots, m$ there exist real numbers α^i, β^i and γ^i such that

$$\alpha^i u_i(t) + \beta^i y_i(t) + \gamma^i = 0$$

for $t \in [0, \epsilon)$. The real-analyticity of Bohl functions implies that

$$\alpha^i u_i(t) + \beta^i y_i(t) + \gamma^i = 0$$

for $t \in \mathbb{R}_+$. Hence,

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \operatorname{affn} \mathscr{G}^i_{l_i} \quad \text{for each } i = 1, 2, \ldots, m \text{ and } t \in \mathbb{R}_+.$$

2(a). Define the sets $[\xi^i]_{i=1}^k$

$$\xi^1 = \left\{[z^i]_{i=1}^k \subset \mathbb{R} \mid z^2 = z^3 = \cdots = z^k = 0\right\}, \tag{20a}$$

$$\xi^k = \left\{[z^i]_{i=1}^k \subset \mathbb{R} \mid z^1 = 0,\ z^2 = z^3 = \cdots = z^{k-1} = 1\right\}, \tag{20b}$$

$$\xi^j = \left\{ [z^i]_{i=1}^k \subset \mathbb{R} \,|\, z^i = \begin{cases} 0, & i = 1, \\ 1, & i = 2, 3, \ldots, j-1, \\ 0, & i = j+1, j+2, \ldots, k \end{cases} \right\}. \tag{20c}$$

Note that they are similar to ζ^j's as defined in (8a) but without inequalities. Define also the sets $\mathscr{Y}^l$

$$\mathscr{Y}^l = \left\{[z^j]_{j=1}^k \subset \mathbb{R}^m \mid [z_j^i]_{i=1}^k \in \xi^{l_j} \text{ for } j = 1, 2, \ldots, m\right\}.$$

Let $\mathscr{Z}^l$ be defined as in (A.1). It follows from the definition of horizontal complementarity representations that

$$\mathscr{G}_{l_i}^i = \left\{ \begin{pmatrix} q_i^u \\ q_i^y \end{pmatrix} - \begin{pmatrix} R_{ii}^1 \\ S_{ii}^1 \end{pmatrix} z_i^1 + \sum_{j=2}^k \begin{pmatrix} R_{ii}^j \\ S_{ii}^j \end{pmatrix} z_i^j \,\middle|\, [z^j]_{j=1}^k \in \mathscr{Z}^l \right\}. \tag{21}$$

Moreover, it can be verified that

$$\operatorname{affn} \mathscr{G}_{l_i}^i = \left\{ \begin{pmatrix} q_i^u \\ q_i^y \end{pmatrix} - \begin{pmatrix} R_{ii}^1 \\ S_{ii}^1 \end{pmatrix} z_i^1 + \sum_{j=2}^k \begin{pmatrix} R_{ii}^j \\ S_{ii}^j \end{pmatrix} z_i^j \,\middle|\, [z^j]_{j=1}^k \in \mathscr{Y}^l \right\}. \tag{22}$$

Then, Lemma 6.3 item 1 implies that there exist functions $z^j : \mathbb{R} \to \mathbb{R}^m$ such that

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} = \begin{pmatrix} q_i^u \\ q_i^y \end{pmatrix} - \begin{pmatrix} R_{ii}^1 \\ S_{ii}^1 \end{pmatrix} z_i^1(t) + \sum_{j=2}^k \begin{pmatrix} R_{ii}^j \\ S_{ii}^j \end{pmatrix} z_i^j(t), \tag{23a}$$

$$[z^j(t)]_{j=1}^k \in \mathscr{Y}^l \tag{23b}$$

for all $t \in \mathbb{R}_+$. Note that the functions z_i^j with $j \neq l_i$ are constant functions due to the definition of the set $\mathscr{Y}^l$. Define the function $z : \mathbb{R} \to \mathbb{R}^m$, and vectors $\bar{u}^l$ and $\bar{y}^l$ as

$$z = \begin{pmatrix} z_1^{l_1} \\ z_2^{l_2} \\ \vdots \\ z_m^{l_m} \end{pmatrix}, \quad \bar{u}^l = q^u + \begin{pmatrix} \sum_{j=2}^{l_1-1} R_{11}^j \\ \sum_{j=2}^{l_2-1} R_{22}^j \\ \vdots \\ \sum_{j=2}^{l_m-1} R_{mm}^j \end{pmatrix}, \quad \bar{y}^l = q^y + \begin{pmatrix} \sum_{j=2}^{l_1-1} S_{11}^j \\ \sum_{j=2}^{l_2-1} S_{22}^j \\ \vdots \\ \sum_{j=2}^{l_m-1} S_{mm}^j \end{pmatrix}.$$

One can check that (23a), (23b) yields that

$$u = \bar{u}^l + \mathscr{R}^l z, \tag{24a}$$
$$y = \bar{y}^l + \mathscr{S}^l z. \tag{24b}$$

It remains to prove that z is a Bohl function. It follows from (24a) and (24b) that the pair (z, x) satisfies

$$\dot{x} = Ax + B\mathscr{R}^l z + B\bar{u}^l, \tag{25a}$$
$$0 = Cx + (D\mathscr{R}^l - \mathscr{S}^l)z + D\bar{u}^l - \bar{y}^l. \tag{25b}$$

Since $G(s)\mathscr{R}^l - \mathscr{S}^l$ is a column representative of $[G(s)R^j - S^j]_{j=1}^k$, it is invertible as a rational matrix due to the hypothesis. Consequently, Lemma 6.5 implies that

$z = E^l x + o^l$ for some E^l and o^l. This implies together with (25a) that *x* is Bohl and hence so is *z*.

2(b). It has already been shown in the proof of previous item that the function *z* is Bohl. For each $j \in \overline{m}$ define

$$z_j^1 = \begin{cases} z_j & \text{if } l_j = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

$$z_j^i = \begin{cases} 0 & \text{if } l_j < i, \\ z_j & \text{if } l_j = i, \\ 1 & \text{otherwise,} \end{cases} \qquad \text{for } i = 2, 3, \ldots, k-1, \tag{27}$$

$$z_j^k = \begin{cases} z_j & \text{if } l_j = k, \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

where *z* is as in the previous item. Clearly, (23a), (23b) holds. Since (u, x, y) is an initial solution, we know

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \mathscr{G}^i_{l_i}$$

for each $i \in \overline{m}$ and for all sufficiently small *t*. It follows from Fact 3.5 that $[z_i^j(t)]_{j=1}^k \in \zeta^{l_i}$ for all sufficiently small *t*, where the ζ's are defined as in (8a). Consequently, $[z^i]_{i=1}^k$ is initially *k*-complementary.

2(c). The matrices F^l, G^l, v^l, and w^l can be given as

$$F^l = A + B\mathscr{R}^l E^l,$$

$$G^l = \mathscr{R}^l E^l,$$

$$v^l = B\bar{u}^l + B\mathscr{R}^l o^l,$$

$$w^l = \bar{u}^l + \mathscr{R}^l o^l$$

by substituting *z* into (24b) and (25a).

2(d). From the previous item, it is known that *x* satisfies

$$\dot{x} = F^l x + v^l \tag{29}$$

for some $F^l \in \mathbb{R}^{n \times n}$ and $v^l \in \mathbb{R}^n$. Since *x* is continuous, it is bounded on every finite interval $[0, T]$. It follows from (29) that $\dot{x}$ is also bounded on the interval $[0, T]$. Therefore, it is Lipschitz continuous on $[0, T]$ with a Lipschitz constant depending on only *l* and *T*. $\square$

Our next aim is to get a rational characterization of the existence of initial solutions. To this aim, following the footsteps of the characterization of the initial solutions of linear complementarity systems in [14,15], we define the *horizontal* version of the *rational complementarity problem*.

Problem 6.6 (HLCP($q(s), [M^i]_{i=1}^k$)). Given $q(s) \in \mathbb{R}^m(s)$ and $[M^i(s)]_{i=1}^k \subset \mathbb{R}^{m\times m}(s)$, find $[z^i(s)]_{i=1}^k \subset \mathbb{R}^m(s)$ such that the following conditions hold:
1. $M^1(s)z^1(s) = q(s) + \sum_{i=2}^k M^i(s)z^i(s)$.
2. For all $s \in \mathbb{C}$,

$$z^1(s) \perp z^2(s),$$

$$(s^{-1}e - z^i(s)) \perp z^{i+1}(s) \quad \text{for } i = 2, 3, \dots, k.$$

3. For all sufficiently large σ,

$$0 \leqslant z^1(\sigma),$$

$$0 \leqslant z^i(\sigma) \leqslant e\sigma^{-1} \quad \text{for } i = 2, 3, \dots, k-1,$$

$$0 \leqslant z^k(\sigma).$$

Notice that the conditions 3 imply that $z^i(s)$ is strictly proper for $i = 2, 3, \dots, k-1$.

The initial solutions of piecewise linear systems can be characterized by the strictly proper solutions of corresponding HRCPs as stated in the following lemma.

Lemma 6.7. *Consider a piecewise linear system* PLS($A, B, C, D, [\mathscr{G}^i]_{i=1}^m$). *Let* $G(s) = D + C(sI - A)^{-1}B$ *be the transfer matrix of* $\Sigma(A, B, C, D)$ *and* $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ *be a horizontal complementarity representation of the piecewise linear characteristics* $[\mathscr{G}^j]_{j=1}^m$. *The following statements are equivalent*:
1. PLS($A, B, C, D, [\mathscr{G}^i]_{i=1}^m$) *has an initial solution with the initial state* x_0.
2. HRCP($C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k$) *has a strictly proper solution.*

To prove Lemma 6.7, we need the following technical lemma.

Lemma 6.8. *The family of Bohl functions* $[f^i]_{i=1}^n \subset \mathscr{B}^m$ *is initially k-complementary if and only if their Laplace transforms* $[\hat{f}^i(s)]_{i=1}^n \subset \mathbb{R}^m(s)$ *satisfy the following conditions*:
1. *For all* $s \in \mathbb{C}$,

$$\hat{f}^1(s) \perp \hat{f}^2(s),$$

$$(s^{-1}e - \hat{f}^i(s)) \perp \hat{f}^{i+1}(s) \quad \textit{for } i = 2, 3, \dots, k.$$

2. *For all sufficiently large* σ,

$$0 \leqslant \hat{f}^1(\sigma),$$

$$0 \leqslant \hat{f}^i(\sigma) \leqslant e\sigma^{-1} \quad \textit{for } i = 2, 3, \dots, k-1,$$

$$0 \leqslant \hat{f}^k(\sigma).$$

Proof. It follows directly from the initial value theorem of Laplace transformation. □

Proof of Lemma 6.7. $1 \Rightarrow 2$. Let (u, x, y) be an initial solution of PLS. It follows from Lemma 6.3 item 2 that there exist initially k-complementary Bohl functions $[z^j]_{j=1}^k$ such that

$$u = q^u - R^1 z^1 + R^2 z^2 + R^3 z^3 + \cdots + R^k z^k, \tag{30a}$$

$$y = q^y - S^1 z^1 + S^2 z^2 + S^3 z^3 + \cdots + S^k z^k. \tag{30b}$$

Lemma 6.8 implies that the Laplace transforms of $[z^j]_{j=1}^k$, $[\hat{z}^j(s)]_{j=1}^k$ satisfy items 2 and 3 of Problem 6.6. On the other hand, the Laplace transform of (u, y), $(\hat{u}(s), \hat{y}(s))$ satisfies

$$\hat{y}(s) = C(sI - A)^{-1} x_0 + G(s)\hat{u}(s).$$

The last equation together with the Laplace domain versions of (30a) results in

$$[G(s)R^1 - S^1]\hat{z}^1(s) = C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y + \sum_{j=2}^{k}[G(s)R^j - S^j]\hat{z}^j(s).$$

Hence, $[\hat{z}^j(s)]_{j=1}^k$ is a solution of HRCP$(C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y,$ $[G(s)R^j - S^j]_{j=1}^k)$. It is clear that $[\hat{z}^j(s)]_{j=1}^k$ is strictly proper since these functions are Laplace transforms of Bohl functions.

$2 \Rightarrow 1$. Let $[\hat{z}^j(s)]_{j=1}^k$ be a strictly proper solution of HRCP$(C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k)$. Let $[z^j]_{j=1}^k$ denote the inverse Laplace transform of $[\hat{z}^j(s)]_{j=1}^k$. Define

$$u = q^u - R^1 z^1 + R^2 z^2 + R^3 z^3 + \ldots + R^k z^k,$$
$$y = q^y - S^1 z^1 + S^2 z^2 + S^3 z^3 + \ldots + S^k z^k.$$

Since $[\hat{z}^j(s)]_{j=1}^k$ satisfies

$$[G(s)R^1 - S^1]\hat{z}^1(s) = C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y + \sum_{j=2}^{k}[G(s)R^j - S^j]\hat{z}^j(s),$$

the Laplace transform of (u, y), $(\hat{u}, \hat{y})$ satisfies

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + G(s)\hat{u}(s).$$

Define $\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s)$. It can be easily checked that (u, x, y) is an initial solution of PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x_0 where x denotes the inverse Laplace transform of $\hat{x}(s)$. □

In the next step, we want to go from a rational characterization of initial solvability to an algebraic characterization. First, following [14], we establish a connection between HRCPs and parametrized families of HLCPs.

Theorem 6.9. *Consider a piecewise linear system* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$. *Let* $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ *be a horizontal complementarity representation of the piecewise linear characteristics* $[\mathscr{G}^j]_{j=1}^m$. *Then the statements* 1 *and* 3 *are equivalent, and so are the statements* 2 *and* 4.

1. $\mathrm{HRCP}(C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k)$ *is solvable.*
2. $\mathrm{HRCP}(C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k)$ *is uniquely solvable.*
3. $\mathrm{HLCP}(\sigma C(\sigma I - A)^{-1}x_0 + G(\sigma)q^u - q^y, [G(\sigma)R^j - S^j]_{j=1}^k)$ *is solvable for all sufficiently large* σ.
4. $\mathrm{HLCP}(\sigma C(\sigma I - A)^{-1}x_0 + G(\sigma)q^u - q^y, [G(\sigma)R^j - S^j]_{j=1}^k)$ *is uniquely solvable for all sufficiently large* σ.

Proof. $1 \Leftrightarrow 3$: It follows from Remark 4.9 and [14, Theorem 4.1].

$2 \Leftrightarrow 4$: It follows from Remark 4.9 and [14, Corollary 4.10]. □

In the sequel, we will be dealing with systems having *low index* in the sense as it will be defined in the following.

Definition 6.10. A rational matrix $M(s) \in \mathbb{R}^{m\times m}(s)$ is said to be of *index k* if it is invertible as a rational matrix and $s^{-k}M^{-1}(s)$ is proper rational.

The notion of index will be generalized to families of matrices via column representatives in what follows.

Definition 6.11. A family of rational matrices $[M^i(s)]_{i=1}^k$ is said to be of *index k* if all its column representative matrices are of index *k*.

It is already known from Lemma 6.7 that strictly proper solutions of HRCP play a key role in the analysis of initial solutions. The following theorem establishes an equivalence between the existence of a strictly proper solution of an HRCP and solvability of an HLCP under the assumption that HRCP is uniquely solvable.

Theorem 6.12. *Consider a piecewise linear system* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$. *Let* $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ *be a horizontal complementarity representation of the piecewise linear characteristics* $[\mathscr{G}^j]_{j=1}^m$. *Suppose that* $[G(\sigma)R^j - S^j]_{j=1}^k$ *has the column* $\mathscr{W}$*-property for all sufficiently large* σ. *Then the following statements hold*:

1. *Assume that* $[G(s)R^j - S^j]_{j=1}^k$ *is of index* 1. *The following two statements are equivalent*:
   (a) $\text{HRCP}(C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k)$ *has a strictly proper solution.*
   (b) $\text{HLCP}(Cx_0 + Dq^u - q^y, [DR^j - S^j]_{j=1}^k)$ *has a solution.*
2. *If* $[DR^1 - S^1, DR^k - S^k]$ *is nondegenerate then* $\text{HRCP}(C(sI - A)^{-1}x_0 + s^{-1} G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k)$ *has a strictly proper solution for all initial states* x_0.

Proof. 1(a) $\Rightarrow$ 1(b). Let $[z^j(s)]_{j=1}^k$ be a strictly proper solution of $\text{HRCP}(C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k)$, i.e., $[z^j(s)]_{j=1}^k$ satisfies the items 2 and 3 of Problem 6.6, and

$$(G(s)R^1 - S^1)z^1(s) = C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y + \sum_{j=2}^{k}(G(s)R^j - S^j)z^j(s) \tag{31}$$

for all $s \in \mathbb{C}$. Define $[\bar{z}^j]_{j=1}^k = \lim_{s\to\infty}[sz^j(s)]_{j=1}^k$. It follows from the items 2 and 3 of Definition 6.6 that $[\bar{z}^j]_{j=1}^k$ is k-horizontal complementary. By multiplying (31) by s and letting s tend to ∞, we get

$$(DR^1 - S^1)\bar{z}^1 = Cx_0 + Dq^u - q^y + \sum_{j=2}^{k}(DR^j - S^j)\bar{z}^j.$$

Consequently, $[\bar{z}^j]_{j=1}^k$ is a solution of $\text{HLCP}(Cx_0 + Dq^u - q^y, [DR^j - S^j]_{j=1}^k)$.

1(b) $\Rightarrow$ 1(a). Observe that we have the following two facts.

(i) Since $[G(\sigma)R^j - S^j]_k^{j=1}$ has the column $\mathscr{W}$-property for all sufficiently large σ, $\text{HLCP}(C(\sigma I - A)^{-1}x_0 + \sigma^{-1}G(\sigma)q^u - \sigma^{-1}q^y, [G(\sigma)R^j - S^j]_{j=1}^k)$ is uniquely solvable for all sufficiently large σ. Hence, it follows from Lemma 6.9 that $\text{HRCP}(C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k)$ has a unique solution, say $[z^j(s)]_{j=1}^k$. Clearly, $[\sigma z^j(\sigma)]_{j=1}^k$ is a solution of $\text{HLCP}(\sigma C(\sigma I - A)^{-1}x_0 + G(\sigma)q^u - q^y, [G(\sigma)R^j - S^j]_{j=1}^k)$ for all sufficiently large σ.

(ii) Let $[\bar{z}^j]_{j=1}^k$ be a solution of $\text{HLCP}(Cx_0 + Dq^u - q^y, [DR^j - S^j]_{j=1}^k)$. Clearly, it is also a solution of $\text{HLCP}(Cx_0 + Dq^u - q^y + \sum_{j=1}^k H^j(\sigma)\bar{z}^j, [G(\sigma)R^j - S^j]_{j=1}^k)$ where $H^1(s) = (G(s) - D)R^1$ and $H^j(s) = (D - G(s))R^j$ for $j = 2, 3, \ldots, k$.

By using the Lipschitzian property of solutions of the HLCP (see Appendix A, Lemma A.2 and Theorem A.1) and the triangle inequality, we get

$$\|[\sigma z^j(\sigma) - \bar{z}^j]_{j=1}^k\| \leqslant \alpha\sigma\Bigg(\|C(\sigma I - A)^{-1}x_0 - \sigma^{-1}Cx_0\| + \|G(\sigma)q^u - Dq^u\| + \sum_{j=2}^{k}\|H^j(\sigma)\|\|\bar{z}^j\|\Bigg)$$

for all sufficiently large σ. Note that the right-hand side of this inequality converges to a constant term as σ tends to infinity. This implies that $[z^j(s)]_{j=1}^k$ is strictly proper.

2. Suppose that $[DR^1 - S^1, DR^k - S^k]$ is nondegenerate but the solution of HRCP($C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y, [G(s)R^j - S^j]_{j=1}^k$), $[z^j(s)]_{j=1}^k$ is not strictly proper for some x_0. This means that $[z^1(s), z^k(s)]$ is not strictly proper since $[z^j(s)]_{j=2}^{k-1}$ is strictly proper by definition of Problem 6.6. Let l be an integer such that $\lim_{s\to\infty} s^{-l}[z^1(s), z^k(s)] = [\bar{z}^1, \bar{z}^k] \neq 0$. Clearly, $l \geqslant 0$. Note that $[z^1(s), z^k(s)]$ is a solution of

$$\mathrm{HRCP}\Bigg(C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y + \sum_{j=2}^{k-1}(G(s)R^j - S^j)z^j(s), [G(s)R^1 - S^1, G(s)R^k - S^k]\Bigg).$$

Hence, $\sigma^{-l}[z^1(\sigma), z^k(\sigma)]$ is a solution of

$$\mathrm{HLCP}\Bigg(\sigma^{-l}C(\sigma I - A)^{-1}x_0 + \sigma^{-l-1}G(\sigma)q^u - \sigma^{-l-1}q^y + \sigma^{-l}\sum_{j=2}^{k-1}(G(\sigma)R^j - S^j)z^j(\sigma), [G(\sigma)R^1 - S^1, G(\sigma)R^k - S^k]\Bigg).$$

Since $[z^j(s)]_{j=2}^{k-1}$ is strictly proper, it follows that $[\bar{z}^1, \bar{z}^k]$ is a solution of HLCP(0, $[DR^1 - S^1, DR^k - S^k]$). Then, we have

$$(DR^1 - S^1)\bar{z}^1 = (DR^k - S^k)\bar{z}^k. \tag{32}$$

Note that $(\bar{z}^1)^{\mathrm{T}}\bar{z}^k = 0$. Define the index sets J, K as $J = \{j \mid \bar{z}^1_j \neq 0\}$ and $K = \{j \mid j \notin J\}$. Eq. (32) can be written as

$$\big((DR^1 - S^1)_{\cdot J}(DR^k - S^k)_{\cdot K}\big)\begin{pmatrix}\bar{z}^1_J \\ -\bar{z}^k_K\end{pmatrix} = 0.$$

Note that the matrix on the left-hand side is a column representative of $[DR^1 - S^1, DR^k - S^k]$ and hence nonsingular by the hypothesis. Then, $\bar{z}^1 = \bar{z}^2 = 0$ which contradicts the definition of the integer l. □

7. Global solutions

In this section, we will use the initial solution concept to analyze *global* solutions of PLS. To do so, we need to introduce piecewise Bohl functions. As one can expect from their definition, Bohl functions are related to linear constant coefficient homogeneous differential equations and hence linear (time-invariant) dynamical systems. In our treatment of piecewise linear dynamical systems, piecewise Bohl functions play a similar role to the one played in the study of linear systems by Bohl functions. A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be a *piecewise Bohl function* if for each $t \in \mathbb{R}_+$ there exist a real number $\epsilon > 0$ and a Bohl function g such that $f|_{[t,t+\epsilon)} = g|_{[0,\epsilon)}$. The set of all such functions is denoted by $\mathscr{PB}$. Note that $\mathscr{PB}$ is not closed under time reversal.

The following definition will make clear what is understood by a global solution of $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$.

Definition 7.1. A triple $(u, x, y) \in \mathscr{PB}^{m+n+m}$ is said to be a *solution* on $[0, T)$ of $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x_0 if the following conditions hold for all $t \in [0, T)$:

$$x(t) = x_0 + \int_0^t [Ax(s) + Bu(s)]\,\mathrm{d}s, \tag{33}$$

$$y(t) = Cx(t) + Du(t), \tag{34}$$

$$\begin{pmatrix} u_i(t) \\ y_i(t) \end{pmatrix} \in \mathscr{G}^i \quad \text{for } i = 1, 2, \ldots, k. \tag{35}$$

We can now present the main result of this paper. The theorem below provides both a condition for existence of a unique solution for a given initial state and a condition for existence and uniqueness of solutions for all initial states.

Theorem 7.2. *Consider a piecewise linear system* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$. *Let* $G(s) = D + C(sI - A)^{-1}B$ *be the transfer matrix of* $\Sigma(A, B, C, D)$ *and also let* $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ *be a horizontal complementarity representation of the piecewise linear characteristics* $[\mathscr{G}^j]_{j=1}^m$. *Suppose that* $[G(\sigma)R^j - S^j]_{j=1}^k$ *has the column* $\mathscr{W}$*-property for all sufficiently large* σ. *Then, the following statements hold*:

1. *Assume that* $[G(s)R^j - S^j]_{j=1}^k$ *is of index* 1. *There exists a unique solution on* $[0, \infty)$ *of* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ *with the initial state* x_0 *if and only if* $\mathrm{HLCP}(Cx_0 + Dq^u - q^y, [DR^j - S^j]_{j=1}^k)$ *is solvable.*
2. *If* $[DR^1 - S^1, DR^k - S^k]$ *is nondegenerate then there exists a unique solution on* $[0, \infty)$ *of* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ *for all initial states.*

To prove this theorem, we will utilize the following lemma.

Lemma 7.3. *Consider a piecewise linear system* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$. *Let* $G(s) = D + C(sI - A)^{-1}B$ *be the transfer matrix of* $\Sigma(A, B, C, D)$ *and also let* $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ *be a horizontal complementarity representation of the piecewise linear characteristics* $[\mathscr{G}^j]_{j=1}^m$. *Suppose that* $[G(\sigma)R^j - S^j]_{j=1}^k$ *has the column* $\mathscr{W}$*-property for all sufficiently large* σ. *Suppose also that the set*

$$\mathscr{R} = \left\{x_0 \in \mathbb{R}^n \mid \mathrm{HRCP}(q_{x_0}(s), [G(s)R^i - S^i]_{i=1}^k) \textit{ has a strictly proper solution}\right\}$$

is closed, where $q_{x_0}(s) = C(sI - A)^{-1}x_0 + s^{-1}G(s)q^u - s^{-1}q^y$. *Then, there exists a unique solution on* $[0, \infty)$ *of* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ *with the initial state* x_0 *if and only if* $x_0 \in \mathscr{R}$.

Proof. *If*: Let the initial state $\bar{x}$ be given such that $\bar{x} \in \mathscr{R}$. Hence, it follows from Theorem 6.12 and Lemma 6.7 that $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ has an initial solution with the initial state $\bar{x}$. Let $(u^{\bar{x}}, x^{\bar{x}}, y^{\bar{x}})$ denote this initial solution. We define $\iota : \mathbb{R}^n \to \overline{k}^m$ as

$$\iota(\bar{x}) = l,$$

where l is as in Lemma 6.3 item 1 for the initial solution $(u^{\bar{x}}, x^{\bar{x}}, y^{\bar{x}})$, $\tau : \mathbb{R}^n \to \mathbb{R}$ as

$$\tau(\bar{x}) = \sup \left\{ T \;\middle|\; \begin{pmatrix} u_j^{\bar{x}}(t) \\ y_j^{\bar{x}}(t) \end{pmatrix} \in \mathscr{G}_{\iota(\bar{x})_j} \text{ for all } j \in \overline{m} \text{ and } t \in [0, T] \right\},$$

and $\kappa : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\kappa(\bar{x}) = x^{\bar{x}}(\tau(\bar{x})).$$

Note that $t \mapsto (u^{\bar{x}}, x^{\bar{x}}, y^{\bar{x}})(t + \rho)$ forms an initial solution of $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state $x^{\bar{x}}(\rho)$ whenever $\rho \in [0, \tau(\bar{x}))$. Hence, we have $x^{\bar{x}}(\rho) \in \mathscr{R}$ for all $\rho \in [0, \tau(\bar{x}))$. It follows from the closedness of the set $\mathscr{R}$ and continuity of $x^{\bar{x}}$ that $\kappa(\bar{x}) \in \mathscr{R}$.

Existence: Define $x_{i+1} = \kappa(x_i)$ for $i = 0, 1, \ldots$ From the previous discussion, we know that $x_i \in \mathscr{R}$ and hence $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ admits initial solutions for all initial states x_i due to Lemma 6.7. Let $(u^{x_i}, x^{x_i}, y^{x_i})$ denote an initial solution with the initial state x_i. Define $\tau_k = \sum_{i=1}^k \tau(x_{k-1})$ for $k > 0$ and $\tau_0 = 0$. Also define

$$(u, x, y)|_{[\tau_k, \tau_{k+1}]} = (u^{x_k}, x^{x_k}, y^{x_k})|_{[0, \tau(x^k)]}.$$

It can be verified that (u, x, y) is a solution on $[0, T)$ for some $T > 0$ of $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x_0. Suppose that T is such that there is no solution on $[0, T')$ whenever $T' > T$. However, Lemma 6.3 item 2(c) implies that x is Lipschitz continuous with the Lipschitz constant $\max_{l \in \overline{k}^m} \alpha^l$, where α^l is as in the same item. Hence, x is uniformly continuous on $[0, T)$ and $x^* := \lim_{t \to T^-} x(t)$

exists due to [23, exercise 4.13]. Since $x(t) \in \mathscr{R}$ for all $t \in [0, T)$ and x is continuous, $x^* \in \mathscr{R}$ which means one can extend the solution (u, x, y) beyond $[0, T)$ by using the initial solution of PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x^*. This contradicts the definition of T. Thus, we can conclude that there exist a solution on $[0, \infty)$ of PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x_0.

Uniqueness: Let $(u^i, x^i, y^i) \in \mathscr{PB}^{m+n+m}$ for $i = 1, 2$ denote two solutions of PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x_0. Clearly, $(u^1, x^1, y^1) - (u^2, x^2, y^2)$ is a piecewise Bohl function as well. If it is not identically zero then there should exist $t \geqslant 0$ and $\epsilon > 0$ such that $((u^1, x^1, y^1) - (u^2, x^2, y^2))|_{[0,t]} \equiv 0$ and $((u^1, x^1, y^1) - (u^2, x^2, y^2))(s) \neq 0$ for all $s \in (t, t + \epsilon)$ due to the definition of piecewise Bohl functions. For (u^i, x^i, y^i) and $t \geqslant 0$, one can find $\epsilon_i > 0$ and Bohl functions $(\bar{u}^i, \bar{x}^i, \bar{y}^i)$ such that $(u^i, x^i, y^i)|_{[t,t+\epsilon_i)} = (\bar{u}^i, \bar{x}^i, \bar{y}^i)|_{[0,\epsilon_i)}$ with $i = 1, 2$ again by the definition of piecewise Bohl functions. It is easy to see that $(\bar{u}^i, \bar{x}^i, \bar{y}^i)$ form two different initial solutions of PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the same initial state, $x^1(t) = x^2(t)$. Then, Lemma 6.7 and Theorem 6.12 imply that HLCP$(C(\sigma I - A)^{-1}x^1(t) + s^{-1}G(\sigma)q^u - s^{-1}q^y, [G(\sigma)R^j - S^j]_{j=1}^k)$ has at least two different solutions for all sufficiently large σ which is ruled out by Theorem 4.8 since $[G(\sigma)R^j - S^j]_{j=1}k$ has the column $\mathscr{W}$-property for all sufficiently large σ.

Only if: Let $(u, x, y) \in \mathscr{PB}^{m+n+m}$ be the unique solution of PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x_0. By the definition of piecewise Bohl functions, we know that there exist $\epsilon > 0$ and $(\bar{u}, \bar{x}, \bar{y}) \in \mathscr{B}^{m+n+m}$ such that $(u, x, y)|_{[0,\epsilon)} = (\bar{u}, \bar{x}, \bar{y})|_{[0,\epsilon)}$. Obviously, $(\bar{u}, \bar{x}, \bar{y})$ is an initial solution of PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ with the initial state x_0. Hence, $x_0 \in \mathscr{R}$ due to Lemma 6.7. $\square$

Proof of Theorem 7.2. 1. Let $\mathscr{R}$ be defined as in Lemma 7.3. It follows from Theorem 6.12 item 1 that

$$\mathscr{R} = \big\{x_0 \mid \text{HLCP}(Cx_0 + Dq^u - q^y, [DR^j - S^j]_{j=1}^k) \\ \text{has a strictly proper solution}\big\}.$$

The set $\mathscr{R}$ is closed since the set $\{q \in \mathbb{R}^n \mid \text{HLCP}(q, [M^i]_{i=1}^k) \text{ is solvable}\}$ is closed. Then, Lemma 7.3 proves the statement.

2. Let $\mathscr{R}$ be defined as in Lemma 7.3. It follows from Theorem 6.12 item 2 that $\mathscr{R} = \mathbb{R}^n$. Then, Lemma 7.3 proves the statement. $\square$

Notice that the horizontal complementarity representations of a family of piecewise linear characteristics are not unique in general. However, the (sufficient) condition presented above for well-posedness depends on those representations. Naturally, one might ask whether it is possible that the condition holds for one representation but not for another one. As stated in the following theorem, the answer to this question is negative. In other words, the above theorem is independent of the choice of the representations.

Theorem 7.4. *Consider a matrix pair* $(M, N) \in \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times m}$ *and k-piecewise linear characteristics* $[\mathscr{G}^j]_{j=1}^m$. *Let* $(\cdot, \cdot, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ *and* $(\cdot, \cdot, [\bar{R}^j]_{j=1}^k, [\bar{S}^j]_{j=1}^k)$ *be horizontal complementarity representations of* $[\mathscr{G}^j]_{j=1}^m$. *If* $[MR^j + NS^j]_{j=1}^k$ *has the column* $\mathscr{W}$*-property then so does* $[M\bar{R}^j + N\bar{S}^j]_{j=1}^k$.

To construct a proof of Theorem 7.4, we need some preparations. Three rather technical lemmas on piecewise linear characteristics are in order. The first one presents equivalent conditions for redundancy of a vertex.

Lemma 7.5. *Let* $\mathscr{G} = \mathrm{plc}(d^-, [v^i]_{i=1}^{k-1}, d^+)$ *be a k-piecewise linear characteristic and* $\mathscr{G}_i$ *be as in Definition* 3.1 *for* $i \in \overline{k}$. *Also let the vectors r and s be defined for the piecewise characteristic* $\mathscr{G}$ *in accordance with* (10a) *and* (10b). *The following statements are equivalent*:
1. *The vertex* v^i *is redundant.*
2. *The set* $\mathscr{G}_i \cup \mathscr{G}_{i+1}$ *is convex.*
3. *There exists* $\alpha > 0$ *such that* $r_j = \alpha r_{j+1}$ *and* $s_j = \alpha s_{j+1}$.

Proof. $1 \Leftrightarrow 2$: Evident.

$2 \Rightarrow 3$: We prove the statement only for $1 \neq i \neq k-1$. The other two cases can be proven in a similar fashion. Note that $v^i = \mathscr{G}_i \cap \mathscr{G}_{i+1}$ and $\mathscr{G}_i = \{\lambda v^{i-1} + (1-\lambda)v^i \mid 0 \leqslant \lambda \leqslant 1\}$. Since $\mathscr{G}_i \cup \mathscr{G}_{i+1}$ is convex, we can conclude that $\mathscr{G}_i \cup \mathscr{G}_{i+1} = \{\lambda v^{i-1} + (1-\lambda)v^{i+1} \mid 0 \leqslant \lambda \leqslant 1\}$. By writing v^i as a convex combination of v^{i-1} and v^{i+1}, we get

$$v^i = \lambda v^{i-1} + (1-\lambda)v^{i+1}.$$

Hence,

$$\underbrace{v^i - v^{i-1}}_{\begin{pmatrix} r_i \\ s_i \end{pmatrix}} = \underbrace{\frac{1-\lambda}{\lambda}}_{>0}(\underbrace{v^{i+1} - v^i}_{\begin{pmatrix} r_{i+1} \\ s_{i+1} \end{pmatrix}}).$$

$3 \Rightarrow 2$. It is enough to show that v^i can be written as the convex combination of v^{i-1} and v^{i+1}. Since there exists $\alpha > 0$ such that $r_j = \alpha r_{j+1}$ and $s_j = \alpha s_{j+1}$, we get

$$v^i - v^{i-1} = \alpha(v^{i+1} - v^i).$$

It follows that

$$v^i = \frac{1}{1+\alpha}v^{i+1} + \frac{\alpha}{1+\alpha}v^{i-1}.$$

Note that $0 \leqslant 1/(1+\alpha) \leqslant 1$. $\square$

By utilizing these properties of redundant vertices, it can be shown that arbitrary descriptions of a piecewise linear characteristic must have some common properties in terms of minimal descriptions as stated in the following lemma.

Lemma 7.6. *Let* $\mathscr{G} = \mathrm{plc}(d^-, [v^i]_{i=1}^{k-1}, d^+)$ *be a k-piecewise linear characteristic and* $(d^-, [v^{l_i}]_{i=1}^{k'-1}, d^+)$ *be one of its minimal descriptions. Also let the vector pairs* (r, s) *and* $(r^{\min}, s^{\min})$ *be defined for* $\mathrm{plc}(d^-, [v^i]_{i=1}^{k-1}, d^+)$ *and* $\mathrm{plc}(d^-, [v^{l_i}]_{i=1}^{k'-1}, d^+)$ *in accordance with* (10a) *and* (10b), *respectively. Then, the following statements hold*:

1. *For each* $j \in \overline{k}$ *there exist* $\alpha > 0$ *and* $p \in \overline{k'}$ *such that* $r_j = \alpha r_p^{\min}$ *and* $s_j = \alpha s_p^{\min}$.
2. *For each* $p \in \overline{k'}$ *there exist* $\alpha > 0$ *and* $j \in \overline{k}$ *such that* $r_p^{\min} = \alpha r_j$ *and* $s_p^{\min} = \alpha s_j$.

Proof. All the statements that will be made for the vectors r and $r^{\min}$ are equally valid for the vectors s and $s^{\min}$ in the rest of this proof.

1. We distinguish four cases.

- *Case* 1: $j \in \{1, k\}$. Obviously,

$$p = \begin{cases} 1 & \text{if } j = 1, \\ k' & \text{if} j = k. \end{cases}$$

  and $\alpha = 1$ do the job.
- *Case* 2: $j \in \{2, 3, \ldots, l_1\}$. Note that $v^{j'}$ is redundant for all $j' \in \{1, 2, \ldots, l_1 - 1\}$. It follows from Lemma 7.5 that there exists $\alpha_{j'}$ such that $r_{j'+1} = \alpha_{j'} r_{j'}$. Therefore, $r_j = (\prod_{j'=1}^{j-1} \alpha_{j'}) r_1$. Consequently, $p = 1$ and $\alpha = \prod_{j'=1}^{j-1} \alpha_{j'}$ do the job.
- *Case* 3: $j \in \{l_{p-1} + 1, l_{p-1} + 2, \ldots, l_p\}$. Note that $v^{j'}$ is redundant for all $j' \in \{l_{p-1} + 1, l_{p-1} + 2, \ldots, l_p - 1\}$. It follows from Lemma 7.5 that there exists $\alpha_{j'}$ such that $r_{j'+1} = \alpha_{j'} r_{j'}$. Thus, we get

$$r_{j''} = \begin{cases} (\prod_{j'=j''}^{j-1} \alpha_{j'})^{-1} r_j & \text{if } j'' \in \{l_{p-1}, l_{p-1} + 1, \ldots, j - 1\}, \\ r_j & \text{if } j'' = j, \\ (\prod_{j'=j}^{j''-1} \alpha_{j'}) r_j & \text{if } j'' \in \{j + 1, j + 2, \ldots, l_p\}. \end{cases} \tag{36}$$

  On the other hand, we have

$$\begin{aligned} r_p^{\min} &= {}_1^{l_p} - v_1^{l_{p-1}} \\ &= \underbrace{v_1^{l_p} - v_1^{l_p - 1}}_{r_{l_p}} + \underbrace{v_1^{l_p - 1} - v_1^{l_p - 2}}_{r_{l_p - 1}} - \cdots + \underbrace{v_1^{l_{p-1}+1} - v_1^{l_{p-1}}}_{r_{l_{p-1}+1}}. \end{aligned}$$

  By using (36), we get

$$r_p^{\min} = \beta r_j$$

  for some $\beta > 0$. Therefore, p and $\alpha = 1/\beta$ do the job.

- *Case* 4: $j \in \{l_{k'-1}+1, l_{k'-1}+2, \dots, k-1\}$. Note that $v^{j'}$ is redundant for all $j' \in \{l_{k'-1}+1, l_{k'-1}+2, \dots, k-1\}$. It follows from Lemma 7.5 that there exists $\alpha_{j'}$ such that $r_{j'} = \alpha_{j'} r_{j'+1}$. Therefore, $r_j = (\prod_{j'=j}^{k-1} \alpha_{j'}) r_k$. Consequently, $p = k$ and $\alpha = \prod_{j'=j}^{k-1} \alpha_{j'}$ do the job.

2. The proof of the previous item shows that one can find positive α's for the following choices of j's.

- *Case* 1: $p \in \{1, k'\}$. Take $j = 1$ if $p = 1$ and $j = k$ if $p = k'$.
- *Case* 2: $p \in \{2, 3, \dots, k'-1\}$. Take any $j \in \{l_{p-1}+1, l_{p-1}+2, \dots, l_p\}$. $\square$

As indicated in Remark 3.2, there are exactly two minimal descriptions. The following lemma depicts how those descriptions are related to each other.

Lemma 7.7. *Let* $(d^-, v^1, v^2, \dots, v^{k-1}, d^+)$ *be a minimal description of a k-piecewise linear characteristic* $\mathscr{G}$. *Also let the vector pairs* $(r^{\min}, s^{\min})$ *and* $(r^{\min'}, s^{\min'})$ *be defined for* $\operatorname{plc}(d^-, v^1, v^2, \dots, v^{k-1}, d^+)$ *and* $\operatorname{plc}(d^+, v^{k-1}, v^{k-2}, \dots, v^1, d^-)$ *in accordance with* (10a) *and* (10b), *respectively. Then,* $r_j^{\min} = r_{k+1-j}^{\min'}$ *and* $s_j^{\min} = s_{k+1-j}^{\min'}$ *for each* $j \in \overline{k}$.

Proof. Evident. $\square$

Finally, we can prove Theorem 7.4 by employing the above lemmas.

Proof of Theorem 7.4. Let $\operatorname{plc}(d^{i,-}, [v^{i,j}]_{j=1}^{k-1}, d^{i,+})$ and $\operatorname{plc}(\bar{d}^{i,-}, [\bar{v}^{i,j}]_{j=1}^{k-1}, \bar{d}^{i,+})$ for $i \in \overline{m}$ be the descriptions of the piecewise linear characteristics $[\mathscr{G}^i]_{i=1}^m$ corresponding to the horizontal complementarity representations $(\cdot, \cdot, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ and $(\cdot, \cdot, [\bar{R}^j]_{j=1}^k, [\bar{S}^j]_{j=1}^k)$, respectively.

(i) Assume that for each $i \in \overline{m}$ the minimal descriptions of $\operatorname{plc}(d^{i,-}, [v^{i,j}]_{j=1}^{k-1}, d^{i,+})$ and $\operatorname{plc}(\bar{d}^{i,-}, [\bar{v}^{i,j}]_{j=1}^{k-1}, \bar{d}^{i,+})$ are the same and $(d^{i,-}, [v^{i,l_j}]_{j=1}^{k_i'-1}, d^{i,+})$. Note that every column representative matrix of $[M\bar{R}^j + N\bar{S}^j]_{j=1}^k$ is of the form

$$
\begin{aligned}
([M\bar{R}^j + N\bar{S}^j]_{j=1}^k)^l &= \big((M\bar{R}^{l_1} + N\bar{S}^{l_1})_{\cdot 1} \cdots (M\bar{R}^{l_m} + N\bar{S}^{l_m})_{\cdot m}\big) \\
&= M\operatorname{diag}(\bar{r}_1^{l_1}, \bar{r}_2^{l_2}, \dots, \bar{r}_m^{l_m}) + N\operatorname{diag}(\bar{s}_1^{l_1}, \bar{s}_2^{l_2}, \dots, \bar{s}_m^{l_m})
\end{aligned}
\tag{37}
$$

for some $l \in \overline{k}^m$. However, (37) and Lemma 7.6 imply that for each column representative matrix of $[M\bar{R}^j + N\bar{S}^j]_{j=1}^k$ one can find a column representative matrix of $[MR^j + NS^j]_{j=1}^k$ such that the determinants of these two representative matrices have the same sign. Since $[MR^j + NS^j]_{j=1}^k$ enjoys the column $\mathscr{W}$-property, so does $[M\bar{R}^j + N\bar{S}^j]_{j=1}^k$.

(ii) As already noted in Remark 3.2, there are exactly two minimal descriptions of a k-piecewise linear characteristic. Let q of the minimal descriptions of $\operatorname{plc}(d^{i,-}, [v^{i,j}]_{j=1}^{k-1}, d^{i,+})$ and $\operatorname{plc}(\bar{d}^{i,-}, [\bar{v}^{i,j}]_{j=1}^{k-1}, \bar{d}^{i,+})$ be different. Eq. (37), Lemmas 7.7 and 7.6 imply that for each column representative matrix of $[M\bar{R}^j + N\bar{S}^j]_{j=1}^k$ one can find a column representative matrix of $[MR^j + NS^j]_{j=1}^k$ such that the sign of the determinant of the former is equal to $(-1)^q$ times the sign of the determinant of the latter. Since $[MR^j + NS^j]_{j=1}^k$ enjoys the column $\mathscr{W}$-property, so does $[M\bar{R}^j + N\bar{S}^j]_{j=1}^k$. $\square$

8. Examples

In this section we apply Theorem 7.2 to subclasses of piecewise linear system.

8.1. Linear complementarity systems

The well-posedness results for linear complementarity systems that have been presented in [5,4,13] can be obtained as a special case of Theorem 7.2.

Corollary 8.1. *Consider a piecewise linear system* $\operatorname{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$, *where the piecewise linear characteristic* $\mathscr{G}^i$ *is as depicted in Fig. 7 for each* $i \in \overline{m}$. *Suppose that* $[I, G(s)]$ *is of index* 1 *and* $G(\sigma)$ *is a* $\mathscr{P}$*-matrix for all sufficiently large* σ. *Then, there exists a unique solution on* $[0, \infty)$ *of* $\operatorname{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ *with the initial state* x_0 *if and only if* $Cx_0 \in K(D)$.

Proof. Note that $\mathscr{G}^i = \operatorname{plc}(d^{i,-}, v^{i,1}, d^{i,+})$, where

$$d^{i,-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v^{i,1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad d^{i,+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore,

$$r^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad s^i = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

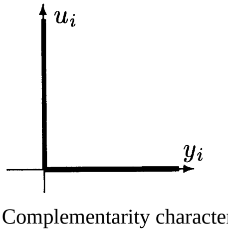

Fig. 7. Complementarity characteristic.

A horizontal complementarity representation $(q^u, q^y, [R^1, R^2], [S^1, S^2])$ of $[\mathscr{G}^i]_m^{i=1}$ can be given by

$$q^u = q^y = 0, \quad R^1 = 0, \quad R^2 = I, \quad S^1 = -I \quad \text{and} \quad S^2 = 0.$$

Hence $[G(s)R^1 - S^1, G(s)R^2 - S^2] = [I, G(s)]$. Note that there is a natural correspondence between the column representative matrices of $[I, G(s)]$ and the principal submatrices of $G(s)$. This fact implies that the ordered matrix set $[I, G(\sigma)]$ has the column $\mathscr{W}$-property for all sufficiently large σ if and only if $G(\sigma)$ is a $\mathscr{P}$-matrix for all sufficiently large σ. Note also that $[DR^1 - S^1, DR^2 - S^2] = [I, D]$. According to Remark 4.9, this implies that HLCP($Cx_0, [I, D]$) is solvable if and only if LCP(Cx_0, D) is solvable. Therefore, the assertion follows from Theorem 7.2 item 1. □

8.2. Linear relay systems

The existence and uniqueness of solutions of linear relay systems (linear systems coupled with relay characteristics) are addressed in [20] (see also [14]). The following corollary states a result that is parallel to those stated in [14,20].

Corollary 8.2. *Consider a piecewise linear system* PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$, *where the piecewise linear characteristic $\mathscr{G}^i$ is as depicted in Fig.* 8 *with $e_2^i > e_1^i$ for each $i \in \overline{m}$. Suppose that $G(\sigma)$ is a $\mathscr{P}$-matrix for all sufficiently large σ. Then, there exists a unique solution on* $[0, \infty)$ *of* PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ *for all initial states x_0.*

Proof. Note that $\mathscr{G}^i = \text{plc}(d^{i,-}, v^{i,1}, v^{i,2}, d^{i,+})$, where

$$d^{i,-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v^{i,1} = \begin{pmatrix} e_1^i \\ 0 \end{pmatrix}, \quad v^{i,2} = \begin{pmatrix} e_2^i \\ 0 \end{pmatrix} \quad \text{and} \quad d^{i,+} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Therefore,

$$r^i = \text{col}(0, e_2^i - e_1^i, 0) \quad \text{and} \quad s^i = \text{col}(-1, 0, -1).$$

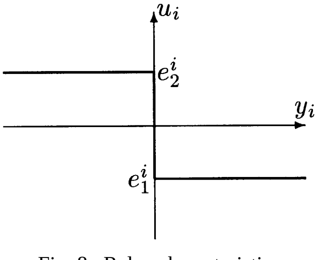

Fig. 8. Relay characteristic.

A horizontal complementarity representation $(q^u, q^y, [R^1, R^2, R^3], [S^1, S^2, S^3])$ of $[\mathscr{G}^i]_{i=1}^m$ can be given by

$$q^u = \mathrm{col}(e_1^1, e_1^2, \ldots, e_1^m), \quad q^y = 0,$$

$$R^1 = R^3 = 0, \quad R^2 = \mathrm{diag}(e_2^1 - e_1^1, e_2^2 - e_1^2, \ldots, e_2^m - e_1^m),$$

$$S^1 = S^3 = -I, \quad S^2 = 0.$$

Hence $[G(s)R^j - S^j]_{j=1}^3 = [I, G(s)R^2, I]$. Since R^2 is a diagonal matrix with positive elements on the diagonal and $[DR^1 - S^1, DR^3 - S^3] = [I, I]$, the following facts can be inferred.

- The ordered matrix set $[I, G(s)R^2, I]$ has the column $\mathscr{W}$-property for all sufficiently large σ if and only if $G(\sigma)$ is a $\mathscr{P}$-matrix for all sufficiently large σ.
- The ordered matrix set $[DR^1 - S^1, DR^3 - S^3]$ is nondegenerate.

The assertion follows immediately from the facts listed above together with Theorem 7.2 item 2. $\square$

We can take one step further and consider relays with deadzone. The next corollary shows that the condition presented for relay systems is also sufficient for the well-posedness of linear systems coupled with relays having deadzone.

Corollary 8.3. *Consider a piecewise linear system* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$, *where the piecewise linear characteristic* $\mathscr{G}^i$ *is as depicted in Fig.* 9 *with* $0 \leqslant e_2^i > e_1^i \leqslant 0$ *and* $f_1^i > f_2^i$ *for each* $i \in \overline{m}$. *Suppose that* $G(\sigma)$ *is a* $\mathscr{P}$*-matrix for all sufficiently large* σ. *Then, there exists a unique solution on* $[0, \infty)$ *of* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ *for all initial states* x_0.

Proof. Note that $\mathscr{G}^i = \mathrm{plc}(d^{i,-}, v^{i,1}, v^{i,2}, v^{i,3}, v^{i,4}, d^{i,+})$, where

$$d^{i,-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v^{i,1} = \begin{pmatrix} e_1^i \\ f_1^i \end{pmatrix}, \quad v^{i,2} = \begin{pmatrix} 0 \\ f_1^i \end{pmatrix}, \quad v^{i,3} = \begin{pmatrix} 0 \\ f_2^i \end{pmatrix},$$

$$v^{i,4} = \begin{pmatrix} e_2^i \\ f_2^i \end{pmatrix} \quad \text{and} \quad d^{i,+} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Therefore,

$$r^i = \mathrm{col}(0, -e_1^i, 0, e_2^i, 0) \quad \text{and} \quad s^i = \mathrm{col}(-1, 0, f_2^i - f_1^i, 0, -1).$$

A horizontal complementarity representation $(q^u, q^y, [R^1, R^2, \ldots, R^5], [S^1, S^2, \ldots, S^5])$ of $[\mathscr{G}^i]_{i=1}^m$ can be given by

$$q^u = \mathrm{col}(e_1^1, e_1^2, \ldots, e_1^m), \quad q^y = \mathrm{col}(f_1^1, f_1^2, \ldots, f_1^m),$$

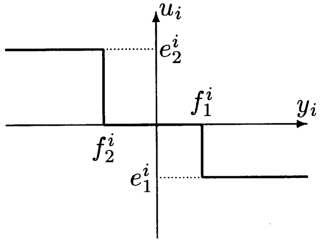

Fig. 9. Relay with deadzone characteristic.

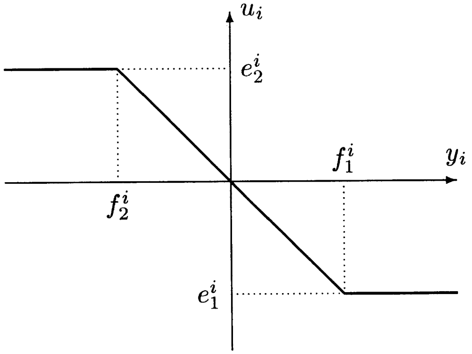

Fig. 10. Saturation characteristic.

$$R^1 = R^3 = R^5 = 0,\ R^2 = -\mathrm{diag}(e_1^1, e_1^2, \ldots, e_1^m),\ R^4 = \mathrm{diag}(e_2^1, e_2^2, \ldots, e_2^m),$$

$$S^1 = S^5 = -I,\ \ S^2 = S^4 = 0,\ \ S^3 = \mathrm{diag}(f_2^1 - f_1^1, f_2^2 - f_1^2, \ldots, f_2^m - f_1^m).$$

Hence $[G(s)R^j - S^j]_{j=1}^5 = [I, G(s)R^2, S^3, G(s)R^4, I]$. Since R^2, S^3 and R^4 are all diagonal matrices with positive elements on the diagonal and $[DR^1 - S^1, DR^5 - S^5] = [I, I]$, the following facts can be inferred.

- The ordered matrix set $[I, G(s)R^2, S^3, G(s)R^4, I]$ has the column $\mathscr{W}$-property for all sufficiently large σ if and only if $G(\sigma)$ is a $\mathscr{P}$-matrix for all sufficiently large σ.
- The ordered matrix set $[DR^1 - S^1, DR^5 - S^5]$ is nondegenerate.

The assertion follows immediately from the facts listed above together with Theorem 7.2 item 2. $\square$

8.3. Linear systems with saturation

As illustrated in Example 2.1, the Lipschitz continuity argument does not work in general for systems with saturation characteristics. The following corollary gives a sufficient condition for the well-posedness of such systems.

Corollary 8.4. *Consider a piecewise linear system* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$, *where the piecewise linear characteristic* $\mathscr{G}^i$ *is as depicted in Fig.* 10 *with* $e_2^i > e_1^i$ *for each* $i \in \overline{m}$. *Let* $R = \mathrm{diag}(e_2^i - e_1^i)$ *and* $S = \mathrm{diag}(f_2^i - f_1^i)$. *Suppose that* $G(\sigma)R - S$ *is a* $\mathscr{P}$*-matrix for all sufficiently large* σ. *Then, there exists a unique solution on* $[0, \infty)$ *of* $\mathrm{PLS}(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$ *for all initial states* x_0.

Proof. Note that $\mathscr{G}^i = \mathrm{plc}(d^{i,-}, v^{i,1}, v^{i,2}, d^{i,+})$, where

$$d^{i,-} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v^{i,1} = \begin{pmatrix} e_1^i \\ f_1^i \end{pmatrix}, \quad v^{i,2} = \begin{pmatrix} e_2^i \\ f_2^i \end{pmatrix} \quad \text{and} \quad d^{i,+} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Therefore,

$$r^i = \mathrm{col}(0, e_2^i - e_1^i, 0) \quad \text{and} \quad s^i = \mathrm{col}(-1, f_2^i - f_1^i, -1).$$

A horizontal complementarity representation $(q^u, q^y, [R^1, R^2, R^3], [S^1, S^2, S^3])$ of $[\mathscr{G}^i]_{i=1}^m$ can be given by

$$q^u = \mathrm{col}(e_1^1, e_1^2, \ldots, e_1^m), \quad q^y = \mathrm{col}(f_1^1, f_1^2, \ldots, f_1^m),$$

$$R^1 = R^3 = 0, \quad R^2 = \mathrm{diag}(e_2^1 - e_1^1, e_2^2 - e_1^2, \ldots, e_2^m - e_1^m),$$

$$S^1 = S^3 = -I, \quad S^2 = \mathrm{diag}(f_2^1 - f_1^1, f_2^2 - f_1^2, \ldots, f_2^m - f_1^m).$$

Hence $[G(s)R^j - S^j]_{j=1}^3 = [I, G(s)R - S, I]$. Note that $[DR^1 - S^1, DR^3 - S^3] = [I, I]$. Then, the following facts can be inferred.

- The ordered matrix set $[I, G(s)R - S, I]$ has the column $\mathscr{W}$-property for all sufficiently large σ if and only if $G(\sigma)R - S$ is a $\mathscr{P}$-matrix for all sufficiently large σ.
- The ordered matrix set $[DR^1 - S^1, DR^3 - S^3]$ is nondegenerate.

The assertion follows immediately from the facts listed above together with Theorem 7.2(2). □

Note that when $D = 0$ the condition that has been presented in Corollary 8.4 is automatically satisfied. Indeed, in this case well-posedness follows from the well-known Lipschitz continuity argument.

Remark 8.5. The condition for a transfer matrix of being a $\mathscr{P}$-matrix has no obvious physical interpretation. It is worthwhile to note that passive systems satisfy this condition. More precisely, if $\Sigma(A, B, C, D)$ is passive (in sense of [30]), (A, B, C) is minimal and $\mathrm{col}(B, D + D^{\mathrm{T}})$ is of full column rank then the transfer matrix $G(s) := D + C(sI - A)^{-1}B$ is positive definite (and hence a $\mathscr{P}$-matrix) for all sufficiently large real s as shown in [7, Lemma 3.3].

9. Conclusions

In this paper we have considered linear systems with piecewise linear characteristics that can be represented using horizontal complementarity variables. We have proposed a solution concept for this class of systems, and we have presented sufficient conditions under which solutions exist and are unique. In particular we have given, under some conditions, a characterization of the set of initial states for which a solution exists.

We have worked with the class of piecewise Bohl functions, which in a sense is tailored for piecewise linear systems without external inputs. The class of piecewise Bohl functions may however be too small for some applications. A recent paper [22] reports existence and uniqueness results in a larger function space for linear systems with a single relay. To provide the same type of results for arbitrary piecewise linear characteristics is an open problem.

The systems considered in this paper are “closed” dynamical systems (i.e., systems without external variables), even though they are constructed with the aid of an “open” linear system and in fact we made extensive use of input/output system theory. Of course it would be of interest to consider piecewise linear systems with additional external variables, such as would be obtained by taking a linear system and connecting some but not all of its inputs and outputs by means of a piecewise linear relation. As an example, the i/o relation $y(t) = \max_{\tau \leqslant t} u(\tau)$ can be realized (assuming proper initialization) by a system that is obtained in this way. More generally, it might be asked which input/output relationships can be realized by means of piecewise linear systems with external variables.

Given that one has established existence and uniqueness of solutions, a natural next question is how to compute these solutions. Numerical procedures may be constructed on the basis of locating the points in time where transfer to another branch of a characteristic takes place, and re-starting the integration with the new data at each such time point (event tracking schemes). When there are many switches between branches this method may become awkward. There are indications that schemes may be devised that will asymptotically (as the time step goes to zero) converge to the true solution, even when no attempt is made to locate the switch times from one branch to another. Such a consistency result has been recently proven under a passivity assumption for systems with ideal diode characteristics [6], and a similar result has been obtained for relay systems in [12]. Extensions to arbitrary piecewise linear systems are currently under investigation.

Appendix A. Some Lipschitzian properties of HLCP

This appendix is devoted to Lipschitzian properties of HLCP. It is known that the solutions of LCP have the upper Lipschitzian property as shown in [8, Theorem 7.2.1]. Moreover, the solution is even a Lipschitz continuous function of the problem data under certain assumptions (see [8, Theorem 7.3.10]). In what follows, we will extend the Lipschitz continuity property to HLCP. We denote $\|\mathrm{col}(z^1, z^2, \dots, z^k)\|$ by $\|[z]_{j=1}^k\|$ for simplicity.

Theorem A.1. *Assume that $[M^i]_{i=1}^k \subset \mathbb{R}^{m\times m}$ has the column $\mathscr{W}$-property. The function $q \mapsto [z^i]_{i=1}^k$, where $[z^i]_{i=1}^k$ is the unique solution of* HLCP$(q, [M^i]_{i=1}^k)$ *is Lipschitz continuous with the Lipschitz constant $d([M^i]_{i=1}^k)$ given by*

$$d([M^i]_{i=1}^k) := \max_{l\in\overline{k}^m} \|\{([M^i]_{i=1}^k)^l\}^{-1}\|.$$

Proof. For $l \in \overline{k}^m$, define the sets $\mathscr{Z}^l \subset \mathscr{H}\mathscr{C}_k^m$ and $\mathscr{Q}^l \subset \mathbb{R}^m$ as

$$\mathscr{Z}^l = \left\{[z^i]_{i=1}^k \in \mathscr{H}\mathscr{C}_k^m \mid [z_j^i]_{i=1}^k \in \zeta^{l_j} \text{ for } j = 1, 2, \dots, m\right\}, \tag{A.1}$$

$$\mathscr{Q}^l = \Bigg\{q \in \mathbb{R}^m \mid q = M^1 z^1 - M^2 z^2 - M^3 z^3 - \cdots - M^k z^k \\ \text{for some } [z^i]_{i=1}^k \in \mathscr{Z}^l\Bigg\}, \tag{A.2}$$

where $[\zeta^i]_{i=1}^k$ is as in Proposition 3.4. Suppose that $[z^i]_{i=1}^k$ is the unique solution of HLCP$(q, [M^i]_{i=1}^k)$ for some $q \in \mathscr{Q}^l$ with $l \in \overline{k}^m$. Then, we have

$$q = M^1 z^1 - M^2 z^2 - M^3 z^3 - \cdots - M^k z^k. \tag{A.3}$$

Define the index sets $K_j = \{i \mid l_i = j\}$ for $j = 1, 2, \dots, k$. It follows that

$$z_{K_j}^i = \begin{cases} 0 & \text{if } i < j \text{ and } i = 1,\\ e & \text{if } i < j \text{ and } i \geqslant 2,\\ 0 & \text{if } i > j. \end{cases} \tag{A.4}$$

since $[z_j^i]_{i=1}^k \in \zeta^{l_j}$. By substituting the above equations into (A.3), we get

$$q = M_{\cdot K_1}^1 z_{K_1}^1 - M_{\cdot K_2}^2 z_{K_2}^2 - M_{\cdot K_3}^3 z_{K_3}^3 - \cdots - M_{\cdot K_k}^k z_{K_k}^k - \sum_{i=2}^{j-1}\sum_{j=3}^{k} M_{\cdot K_j}^i e_{K_j}. \tag{A.5}$$

Consequently,

$$\begin{pmatrix} M_{\cdot K_1}^1 & -M_{\cdot K_2}^2 & -M_{\cdot K_3}^3 & \cdots & -M_{\cdot K_k}^k \end{pmatrix} \begin{pmatrix} z_{K_1}^1 \\ z_{K_2}^2 \\ z_{K_3}^3 \\ \vdots \\ z_{K_k}^k \end{pmatrix} = q + \sum_{i=2}^{j-1}\sum_{j=3}^{k} M_{\cdot K_j}^i e_{K_j}. \tag{A.6}$$

Note that $K_i \cap K_j = \emptyset$ if $i \neq j$ and $\bigcup_{j=1}^{k} K_i = \overline{m}$. It follows from the fact that $[M^i]_{i=1}^k$ has the column $\mathscr{W}$-property that the matrix

$$\begin{pmatrix} M^1_{\cdot K_1} & -M^2_{\cdot K_2} & -M^3_{\cdot K_3} & \cdots & -M^k_{\cdot K_k} \end{pmatrix}$$

is invertible. Hence, (A.6) can be written as

$$\begin{pmatrix} z^1_{\cdot K_1} \\ z^2_{\cdot K_2} \\ z^3_{\cdot K_3} \\ \vdots \\ z^k_{\cdot K_k} \end{pmatrix} = \begin{pmatrix} M^1_{\cdot K_1} & -M^2_{\cdot K_2} & -M^3_{\cdot K_3} & \cdots & -M^k_{\cdot K_k} \end{pmatrix}^{-1} \left(q + \sum_{i=2}^{j-1} \sum_{j=3}^{k} M^i_{\cdot K_j} e_{K_j} \right). \tag{A.7}$$

Eqs. (A.4) and (A.7) imply that the function $q \mapsto [z^j]_{j=1}^k$ is affine on the set $\mathscr{Q}^l$. The column $\mathscr{W}$-property of $[M^i]_{i=1}^k$ implies from Theorem 4.8 that for each $q \in \mathbb{R}^m$ there exists a solution of HLCP(q, $[M^i]_{i=1}^k$), i.e.,

$$\bigcup_{l \in \overline{k}^m} \mathscr{Q}^l = \mathbb{R}^m \tag{A.8}$$

and this solution is unique, i.e.,

$$(\mathscr{Q}^{l^1} \cap \mathscr{Q}^{l^2})^\circ = \emptyset \quad \text{if } l^1 \neq l^2. \tag{A.9}$$

Furthermore, uniqueness of solutions implies that the function $q \mapsto [z^j]_{j=1}^k$ is continuous. Note that it is even Lipschitz continuous on each $\mathscr{Q}^l$ due to (A.7). It follows from the convexity of the sets $\mathscr{Q}^l$, together with (A.8) and (A.9), that the function $q \mapsto [z^j]_{j=1}^k$ is Lipschitz continuous. Then, the claim of the theorem follows since

$$\begin{aligned} &\left\| \begin{pmatrix} M^1_{\cdot K_1} & -M^2_{\cdot K_2} & -M^3_{\cdot K_3} & \cdots & -M^k_{\cdot K_k} \end{pmatrix} \right\| \\ &\quad = \left\| \begin{pmatrix} M^1_{\cdot K_1} & M^2_{\cdot K_2} & M^3_{\cdot K_3} & \cdots & M^k_{\cdot K_k} \end{pmatrix} \right\|. \qquad \square \end{aligned}$$

In particular, we will need to establish lower bounds for certain transfer matrices. The lemma below gives such a bound for low index transfer matrices.

Lemma A.2. *Consider a piecewise linear system* PLS$(A, B, C, D, [\mathscr{G}^i]_{i=1}^m)$. *Let* $(q^u, q^y, [R^j]_{j=1}^k, [S^j]_{j=1}^k)$ *be a horizontal complementarity representation of the* *piecewise linear characteristics* $[\mathscr{G}^j]_{j=1}^k$ *and* $G(s) = D + C(sI - A)^{-1}B$. *Suppose that* $[G(s)R^j - S^j]_{j=1}^k$ *is of index* 1. *Then, there exists a real number* α *such that for all sufficiently large* σ

$$d([G(\sigma)R^j - S^j]_{j=1}^k) \leqslant \alpha\sigma. \tag{A.10}$$

Proof. By hypothesis, we know that $s^{-1}\{([G(s)R^j - S^j]_{j=1}^k)^l\}^{-1}$ is proper for all $l \in \overline{m}^k$. Hence, $\|\{([G(\sigma)R^j - S^j]_{j=1}^k)^l\}^{-1}\| \leqslant \alpha_l\sigma$ for all sufficiently large σ. Clearly, (A.10) holds for $\alpha = \max_{l\in\bar{k}^m} \alpha_l$. $\square$

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